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Optimal Approximation for Functions Prescribed at Equally Spaced Points

H. F. Weinberger [1]

(February 10, 1961)

Explicit upper and lower bounds for the value $F(u)$ of a linear functional F applied to a function $u(x)$ defined on the interval $0 \leq x \leq 1$ are given when u is prescribed at the $N+1$ points i/N, $i=0, \ldots, N$, and a bound for the integral of $u^{[k]2}$ is known. These bounds are optimal in the sense that they are attained for functions satisfying the prescribed conditions. Their computation involves the inversion of a matrix of size $k-1$ rather than N, which means that N is permitted to be large.

1. Introduction

Many problems in numerical analysis can be reduced to approximating the value $F(u)$ of a given linear functional F operating on an unknown element u of a linear vector space. The approximation is to be made in terms of a finite set of data concerning u. Thus, the values $F_1(u), \ldots, F_n(u)$ of N linear functionals acting on u may be given. For example, the $F_i(u)$ may be values of the function u at certain points x_i. If $F(u)$ is the value of u at another point ξ, we have the problem of linear interpolation. If $F(u)$ is an integral of u, we have the problem of numerical quadratures. If $F(u)$ is the value of a derivative, we have numerical differentiation.

It was shown by M. Golomb and the author [2] that in order to obtain a finite interval in which the value $F(u)$ must lie, one must be given the value of at least one nonlinear functional operating on u. The simplest case is that in which one is given a bound for a quadratic functional (u,u). In this case Golomb and the author [2] showed how to obtain the exact interval in which $F(u)$ must lie when the values $F_1(u), \ldots, F_N(u)$, and (u,u) are given. That is, upper and lower bounds for $F(u)$ which are attained for some elements u satisfying the given conditions are found. The construction of these bounds requires the inversion of a matrix depending upon the functionals $F, F_1, \ldots, F_N$, and (u,u).

In this paper we restrict our attention to a very simple case. We deal with a function $u(x)$ of a single variable on the interval $[0,1]$. The given functionals F_i are the values $u(i/N)$ of u at the $N+1$ equally spaced points i/N, $i=0, \ldots, N$. The quadratic functional is taken to be the integral of the square of the k^{th} derivative of u.

We think of the number of points N as large, while the number k of the derivative will usually be small, say two or three. The matrix to be inverted is of size N. By making use of the equal spacing of our points, we shall reduce its inversion to that of a $(k-1)\times(k-1)$ matrix. Thus the problem of obtaining best formulas for interpolation, quadratures, and numerical differentiation is made manageable even when the number of points involved becomes large.

When $F(u)=\int_0^1 u\,dx$, our results yield as special cases the best quadrature formulas of Sard [3] for $k \leq 3$, $N \leq 6$.

2. Approximation Problem

Let the values of the unknown function $u(x)$ be given at the $N+1$ evenly spaced points i/N, $i=0, \ldots, N$. Let M^2 be a given bound for the integral of the kth derivative of u.

$$(u,u)=\int_0^1 u^{[k]2}dx \leq M^2. \tag{2.1}$$

We assume that $N \geq 2k$, $k \geq 2$.

Our problem is to approximate the value $F(u)$ of a certain linear functional F applied to u. According to the theory in footnote 2, this is possible if and only if the functional F is bounded in the norm (2.1) for functions vanishing at the points i/N. That is, we must assume that there is a constant c such that

$$F(v)^2 \leq c\int_0^1 v^{[k]2}dx \tag{2.2}$$

for all k times differentiable functions $v(x)$ such that

$$v\left(\frac{i}{N}\right)=0, \qquad i=0, \ldots, N. \tag{2.3}$$

Any linear combination of pointwise values or integrals of v and its derivatives up to order k-1 will satisfy this condition.

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[2] M. Golomb and H. F. Weinberger, Optimal approximations and error bounds, Symp. on Numer. Approx. (Univ. Wisconsin Press, Madison, Wis., 1959).

[3] A. Sard, Best approximate integration formulas; best approximation formulas, Am. J. Math. 71, 80-91 (1949).

To obtain a best estimate for $F(u)$ we construct an auxiliary function $\bar{u}$ defined by the properties

$$\bar{u}^{[2k]}(x)=0, \qquad 0<x<1, \quad Nx\neq 0,1,\ldots,N \tag{2.4}$$

and

$$\bar{u}\left(\frac{i}{N}\right)=u\left(\frac{i}{N}\right) \qquad i=0,1,\ldots,N,$$

$$\bar{u}^{[l]}(0)=\bar{u}^{[l]}(1)=0 \qquad l=k,\ldots,2k-2. \tag{2.5}$$

The function u and its first $2k$-2 derivatives are continuous, while $\bar{u}^{[2k-1]}$ is allowed to have jump discontinuities at the points i/N.

The Green's function $G(x;\ \xi)$ is defined by the properties

$$\left.\begin{aligned}
&\frac{\partial^{2k}G}{\partial x^{2k}}=0, && 0<x<1,\ Nx\neq N\xi, 0,1,\ldots N,\\
&G\left(\frac{i}{N};\xi\right)=0, && i=0,1,\ldots,N,\\
&\frac{\partial^{l}G}{\partial x^{l}}=0, && \text{at } x=0,1,\ l=k,\ldots 2k-2,\\
&\frac{\partial^{2k-1}G(\xi+0;\xi)}{\partial x^{2k-1}}-\frac{\partial^{2k-1}G(\xi-0;\xi)}{\partial x^{2k-1}}=(-1)^k.
\end{aligned}\right\} \tag{2.6}$$

Again G and its first $2k$-2 derivatives are continuous, while the $(2k-1)$st derivative may have discontinuities at i/N. By integration by parts we find that

$$\bar{u}(\xi)=\sum_{i=0}^{N} g_i(\xi)u\left(\frac{i}{N}\right), \tag{2.7}$$

where

$$g_i(\xi)=(-1)^{k-1}\left\{\frac{\partial^{2k-1}G\left(\frac{i}{N}+0;\xi\right)}{\partial x^{2k-1}}-\frac{\partial^{2k-1}G\left(\frac{i}{N}-0;\xi\right)}{\partial x^{2k-1}}\right\}. \tag{2.8}$$

Also by integration by parts we have

$$u(\xi)-\bar{u}(\xi)=\int_0^1 \{u^{[k]}-\bar{u}^{[k]}\}\,\frac{\partial^k G(x;\xi)}{\partial x^k}\,dx. \tag{2.9}$$

Applying the functional F and using Schwarz's inequality, we find that

$$\begin{aligned}
|F(u)-F(\bar{u})|^2 &\leq \int_0^1 |u^{[k]}-\bar{u}^{[k]}|^2dx\int_0^1\left|F_\xi\left[\frac{\partial^k G(x;\xi)}{\partial x^k}\right]\right|^2dx\\
&=\int_0^1 |u^{[k]}-\bar{u}^{[k]}|^2dx\,F_\xi\{F_\eta[G(\xi;\eta)]\}.
\end{aligned} \tag{2.10}$$

(The symbol $F_\eta\,[G\,(\xi;\,\eta)]$ means that the functional F is applied to G considered as a function of η for fixed ξ. $F_\xi\{F_\eta[G\,(\xi;\eta)]\}$ means that F is then applied to the function $F_\eta[G]$). We have used the property

$$G(\xi;\eta)=\int_0^1 \frac{\partial^k G(\xi;x)}{\partial x^k}\,\frac{\partial^k G(\eta;x)}{\partial x^k}\,dx, \tag{2.11}$$

which follows from integration by parts.

Another integration by parts shows that

$$\int_0^1 \bar{u}^{[k]}\{u^{[k]}-\bar{u}^{[k]}\}dx=0. \tag{2.12}$$

Hence, we can rewrite (2.10) as

$$|F(u)-F(\bar{u})|^2\leq\left\{M^2-\int_0^1 \bar{u}^{[k]2}dx\right\}F_\xi\{F_\eta[G(\xi;\eta)]\}. \tag{2.13}$$

Once $\bar{u}$ and G are found, this inequality provides upper and lower bounds for $F(u)$. These bounds are sharp in the sense that we can construct functions u satisfying (2.1) and having the given values $u(i/N)$ for which the bounds for $F(u)$ are attained.

We write $\bar{u}$ in the form

$$\bar{u}(x)=\sum_{i=-k+1}^{N+k-1} a_i\Delta^{2k}|Nx-i|^{2k-1}. \tag{2.14}$$

The centered difference operator Δ^{2k} is defined by

$$\Delta^{2k}c_l=\sum_{l=-k}^{k}\binom{2k}{k+l}(-1)^{k+l}c_{l+l} \tag{2.15}$$

It is easily seen that $\Delta^{2k}|Nx-i|^{2k-1}$ vanishes for $|Nx-i|\geq k$. Hence the sum in (2.14) has at most $2k$-1 nonzero terms.

Clearly the function (2.14) satisfies (2.4) and has the required continuity properties.

The coefficients a_i are to be determined by the conditons (2.5). Thus we must have

$$\sum_{i=-k+1}^{N+k-1} a_i\Delta^{2k}|j-i|^{2k-1}=u\left(\frac{j}{N}\right), \qquad j=0,1,\ldots,N. \tag{2.16}$$

In order to apply the last line of (2.5), we first use partial summation to write

$$\bar{u}(x)=(-1)^k\sum_{i=-\infty}^{\infty} D^k a_i D^k|Nx-i|^{2k-1}, \tag{2.17}$$

where we have put $a_i=0$ for $i\leq -k$, $i\geq N+k$, and where D^k is the kth forward difference operator:

$$D^k c_i=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}c_{i+l} \tag{2.18}$$

Since $\Delta^{2k}|Nx-i|^{2k-1}=0$ for $|Nx-i|\geq k$, (2.14) and (2.17) are independent of the values of a_i for $i\leq -k$, $i\geq N+k$ when $0\leq x\leq 1$. We note that $D^k|Nx-i|^{2k-1}$

is a polynomial of degree $k-1$ for $Nx \le i$ or $Nx \ge i+k$. Hence the last line of (2.5) will be satisfied if

$$D^k a_i = 0, \qquad i = -k+1, \ldots, -1, N-k+1, \ldots, N-1. \quad (2.19)$$

The boundary value problem (2.5) is thus replaced by the system (2.16), (2.19) of $N+2k-1$ linear equations in the $N+2k-1$ unknowns a_i.

We turn to this problem of matrix inversion.

Remark: We have assumed that $k \ge 2$. The case $k=1$ is easily treated. Let $\bar{u}$ be the broken linear function coinciding with $\bar{u}$ at the points i/N and having its breaks at these points. Let $G(x;\xi)$ be the broken linear function with breaks at ξ and the two neighboring points of the form i/N which vanishes at the points i/N, and whose derivative decreases by -1 at ξ. Then (2.13) gives optimal bounds for $\dot{F}(u)$ when $k=1$.

3. Matrix Inversion

We consider the system of linear equations (2.16), (2.19). Since $\Delta^{2k}|j-i|^{2k-1}$ depends only upon $|j-i|$ and vanishes for $|j-i| \ge k$, (2.16) is a finite difference equation or order $2k-2$. We solve it by means of a system of $2k-2$ independent solutions of the homogeneous equation. To find these solutions, we note that for any number z

$$\sum_{i=-k+1}^{N+k-1} e^{iz}\Delta^{2k}|j-i|^{2k-1} = -2e^{(j-k)z}(e^z-1)^{2k}\left(\frac{d}{dz}\right)^{2k-1}[(e^z-1)^{-1}] \quad (3.1)$$

for $0 \le j \le N$.

We define the polynomial in e^z

$$Q_l(e^z) \equiv 2(-1)^{l+1}e^{-z}(e^z-1)^{l+2}\left(\frac{d}{dz}\right)^{l+1}[(e^z-1)^{-1}] \quad (3.2)$$

It is easily seen to be of degree l with leading coefficient 2.

The Q_l can be generated by the recursion

$$Q_l(y) = (ly+1)Q_{l-1}(y) - y(y-1)Q_{l-1}(y), \qquad Q_0(y) \equiv 2 \quad (3.3)$$

The first few of these polynomials are

$$\left.\begin{aligned} Q_1(y) &= 2(y+1), \\ Q_2(y) &= 2(y^2+4y+1), \\ Q_3(y) &= 2(y^3+11y^2+11y+1), \\ Q_4(y) &= 2(y^4+26y^3+66y^2+26y+1). \end{aligned}\right\} \quad (3.4)$$

The coefficients of Q_{2k-2} are the coefficients of the finite difference equation (2.16). It can be shown by induction that they are symmetric in the sense that

$$Q_l\left(\frac{1}{y}\right) = y^{-l}Q_l(y) \quad (3.5)$$

This means that the zeros of Q_l occur in reciprocal pairs. When l is odd, one of the zeros is -1. When l is even, Q can be written as a polynomial in $(y+y^{-1})$. Thus, the zeros of Q_{2k-2} can be found by solving an equation of degree $k-1$ and a quadratic equation.

It is shown by induction that the coefficients of Q_l are positive and that the zeros of Q_l are real and negative. The zeros of Q_l separate thos of Q_{l+1}.

Let $y_1 < y_2 < \ldots < y_{2k-2} < 0$ be the zeros of Q_{2k-2}:

$$Q_{2k-2}(y_\nu) = 0, \qquad \nu = 1, \ldots, 2k-2. \quad (3.6)$$

Then by (3.7)

$$y_{2k-1-\nu} = \frac{1}{y_\nu}. \quad (3.7)$$

Because of (3.1) and (3.2), the functions $a_i = y_\nu$ satisfy the homogeneous equations corresponding to (2.16). We now define

$$\psi_i = \begin{cases} 0 & i \le k-2. \\ \sum_{\nu=1}^{2k-2} \dfrac{y_\nu^{i+k-2}}{Q'_{2k-2}(y_\nu)} & i \ge -k+2, \end{cases} \quad (3.8)$$

It follows from the Lagrange interpolation formula [4] that the two definitions of ψ_i coincide for $-k+2 \le i \le k-2$. Using (3.1) and (3.2), we find that

$$\sum_{i=-k-1}^{N+k-1} \psi_{i-p}\Delta^{2k}|j-i|^{2k-1} = \delta_{jp}, \qquad j,p = 0, \ldots, N. \quad (3.9)$$

The same equation is satisfied by ψ_{i-p} plus any linear combination of the y_ν^i. We add a linear combination such that the new function satisfies (2.19). Since ψ_{i-p} vanishes for $i \le p+k-2$ these conditions are already satisfied for $i = -k+1, \ldots, -2$. Therefore we add only functions which also satisfy these conditions. The Lagrange interpolation formula [4] shows that such linear combinations are furnished by the functions $\eta_{i+\alpha}$, $\alpha = 1, \ldots, k$, where

$$\eta_i = \sum_{\nu=1}^{2k-2} \frac{y_\nu^{i+k-2}}{(y_\nu-1)^k Q'_{2k-2}(y_\nu)}. \quad (3.10)$$

Note that

$$D^k\eta_i = \psi_i, \qquad i \ge -k+2. \quad (3.11)$$

We let

$$\Gamma_{ip} = \psi_{i-p} - \sum_{\alpha=1}^{k} c_\alpha \eta_{i+\alpha} \quad (3.12)$$

and determine the coefficients c_α in such a way that

[4] J. F. Steffensen, Interpolation (Chelsea Press, New York, N. Y., 1950.

$D_i^k \Gamma_{ip}=0$ for $i=-1,\ N-k+1,\ \ldots,\ N-1$. This gives $c_k=\delta_{0p}$, and the equations

$$\sum_{\alpha=1}^{k-1} c_\alpha \psi_{N-k+\alpha+\beta}=D^k \psi_{N-k+\beta-p} - \delta_{0p}\psi_{N+\beta}, \qquad \beta=1,\ldots,k-1. \quad (3.13)$$

We let $A_{\alpha\beta}$ be the inverse of the symmetric $(k-1)\times(k-1)$ matrix $\psi_{N-k+\alpha+\beta}$ so that [5]

$$\sum_{\beta=1}^{k-1} A_{\alpha\beta}\psi_{N-k+\beta+\gamma}=\delta_{\alpha\gamma}, \qquad \alpha,\gamma=1,\ldots,k-1. \quad (3.14)$$

Then Γ_{ip} is given by

$$\Gamma_{ip}=\psi_{i-p}-\sum_{\alpha,\beta=1}^{k-1} A_{\alpha\beta}\,\eta_{i+\alpha}\,D^k\psi_{N-k-p+\beta} - \delta_{0p}\{\eta_{i+k}-\sum A_{\alpha\beta}\,\eta_{i+\alpha}\psi_{N+\beta}\}, \qquad i=-k+1,\ldots,N+k-1, \qquad p=0,\ldots,N. \quad (3.15)$$

This function satisfies

$$\sum_{i=-k+1}^{N+k-1} \Gamma_{ip}\,\Delta^{2k}|i-j|^{2k-1}=\delta_{jp}, \qquad j,p=0,\ldots,N \quad (3.16)$$

and

$$D_i^k\,\Gamma_{ip}=0, \qquad i=-k+1,\ldots,-1,\ N-k+1,\ldots,N-1; \qquad p=0,\ldots,N. \quad (3.17)$$

The solution of (2.16), (2.19) is given by

$$a_i=\sum_{p=0}^{N} \Gamma_{ip}\,u\left(\frac{p}{N}\right). \quad (3.18)$$

Therefore Γ_{ip} is the inverse matrix for the problem (2.16), (2.19).

Our problem of matrix inversion has thus been reduced to the solution of polynomial equations of degree $k-1$ and two in order to find the y_r and the inversion of the $(k-1)$-dimensional symmetric matrix $\psi_{N-k+\alpha+\beta}$.

Example: We consider the case $k=2$. The zeros of $Q_2(y)$ are

$$y_1=-2-\sqrt{3}, \qquad y_2=-2+\sqrt{3}. \quad (3.19)$$

The function ψ_i defined by (3.10) is

$$\psi_i=\begin{cases} 0 & i\le 0, \\ \dfrac{\sqrt{3}}{12}\,(y_1^{-i}-y_1^{i}) & i\ge 0. \end{cases} \quad (3.20)$$

[5] The fact that $\psi_{N-k+\alpha+\beta}$ is nonsingular follows from the uniqueness of $\bar{u}$ defined by (2.4), (2.5), and the linear independence of the functions $\Delta^{2k}|Nx-i|^{2k-1}$ and $\eta_{i+\alpha}$, $\alpha=1,\ \ldots,\ k$.

Since $k-1=1$, the matrix $A_{\alpha\beta}$ has the single element ψ_N^{-1}. Moreover,

$$D^p\psi_i=-\frac{1}{2}\sqrt{3}(y_1^{-i-1}-y_1^{i+1}), \qquad i\ge 0,$$

$$D^p\psi_{-1}=\frac{1}{2},$$

$$\eta_i=-\frac{1}{72}\sqrt{3}(y_1^{1-i}-y_1^{i-1}). \quad (3.21)$$

Thus (3.17) gives

$$\Gamma_{ip}=\begin{cases} \dfrac{\sqrt{3}(1-y_1^{-2i})(1-y_1^{-2(N-p)})y_1^{i-p}}{12(1-y_1^{-2N})} & -1\le i\le p, \quad 1\le p\le N-1 \\ \dfrac{\sqrt{3}(1-y_1^{-2p})(1-y_1^{-2(N-i)})y_1^{p-i}}{12(1-y_1^{-2N})} & p\le i\le N+1, \quad 1\le p\le N-1 \\ \dfrac{(1-y_1^{-2i})y_1^{i-N}}{12(1-y_1^{-2N})} & i\le N, \quad p=N \\ \dfrac{(1-y_1^{-2(N-i)})y_1^{-i}}{12(1-y_1^{-2N})} & i=0, \quad p=0 \\ \dfrac{4-\sqrt{3}-(4+\sqrt{3})y_1^{-2N}}{12(1-y_1^{-2N})} & \begin{cases} i=-1, & p=0 \\ i=N+1, & p=N \end{cases} \end{cases} \quad (3.22)$$

4. Bounds

We now return to the consideration of section 2. The function $\bar{u}$ is given by

$$\bar{u}(x)=\sum_{\substack{i=-k+1 \\ |Nx-i|<k}}^{N+k-1} a_i\Delta^{2k}|Nx-i|^{2k-1}, \quad (4.1)$$

with

$$a_i=\sum_{j=0}^{N} \Gamma_{ij}\,u\left(\frac{j}{N}\right). \quad (4.2)$$

Thus, we may write

$$\bar{u}\,(x)=\sum_{j=0}^{N} g_j\,(x)\,u\left(\frac{j}{N}\right), \quad (4.3)$$

where

$$g_j\,(x)=\sum_{\substack{i=-k+1 \\ |Nx-i|<k}}^{N+k-1} \Gamma_{ij}\,\Delta^{2k}|Nx-i|^{2k-1} \quad (4.4)$$

is the optimal approximation function corresponding to $u(i/N)=\delta_{ij}$.

In order to find the bounds (2.13) we need the integral of $\bar{u}^{(k)2}$ and $G(x,\xi)$. Integrating by parts we find that

$$\int_0^1 \bar{u}^{(k)2}\,dx=(-1)^k\sum_{j=0}^{N} u\left(\frac{j}{N}\right)[\bar{u}^{(2k-1)}]_{j/N}, \quad (4.5)$$

where $[\ ]_{j/N}$ denotes the discontinuity in the function at j/N, with the convention $\overline{u}^{(2k-1)}(0-)=\overline{u}^{(2k-1)}(1+)=0$. Applying partial summation to (4.1), we find that

$$\overline{u}(x)=\sum_{i=-k+1}^{N+k-1}|Nx-i|^{2k-1}\Delta^{2k}a_i. \tag{4.6}$$

Consequently,

$$\int_0^1 \overline{u}^{(k)2}dx=(-1)^k 2N^{2k-1}(2k-1)!\sum_{i=0}^{N} u\left(\frac{i}{N}\right)\Delta^{2k}a_i$$

$$=(-1)^k 2N^{2k-1}(2k-1)!$$

$$\sum_{i,j=0}^{N} u\left(\frac{i}{N}\right)u\left(\frac{j}{N}\right)\Delta_i^{2k}\Gamma_{ij} \tag{4.7}$$

with the convention that

$$\Delta_i^{2k}\Gamma_{ij}=D^k\Gamma_{ij} \qquad \text{for} \quad i=0,\ i=N. \tag{4.8}$$

We now construct the Green's function $G(x,\xi)$. We begin with a function $H(x,\xi)$ having the proper jump at $x=\xi$ and satisfying the condition that the derivatives of orders $k, \ldots 2k\text{-}2$ vanish at the end points. Let p be any integer satisfying

$$0\le p<N\xi<p+2k-1\le N. \tag{4.9}$$

It follows from the Lagrange interpolation formula (see footnote 4) that the function

$$H(x;\xi)=|x-\xi|^{2k-1}-\sum_{\mu=0}^{2k-1}\left|\frac{p+\mu}{N}-x\right|^{2k-1}b_\mu(\xi), \tag{4.10}$$

where

$$b_\mu(\xi)=\frac{(-1)^{\mu+1}N^{2k-1}}{(2k-1)!}\binom{2k-1}{\mu}$$

$$\left(\xi-\frac{p+\mu}{N}\right)^{-1}\prod_{\nu=0}^{2k-1}\left(\xi-\frac{p+\nu}{N}\right) \tag{4.11}$$

has the property

$$H(x;\xi)\equiv 0 \qquad \text{for} \quad x\le\frac{p}{N},\quad x\ge\frac{p+2k-1}{N}. \tag{4.12}$$

Therefore, the function

$$G(x;\xi)=\frac{(-1)^k}{2(2k-1)!}\left\{H(x;\xi)-\sum_{j=0}^{N}g_j(x)H\left(\frac{j}{N};\xi\right)\right\} \tag{4.13}$$

has all the properties (2.6). Thus, it is the Green's function.[6] We not that the sums in (4.4) and (4.13) involve at most $2k$—1 terms for each x and ξ.

The bounds (2.13) for $F(u)$ are now given explicitly by

$$\left|F(u)-\sum_{i=-k+1}^{N+k-1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^{2k}|Nx-i|^{2k-1})u\left(\frac{j}{N}\right)\right|^2$$

$$\le\frac{(-1)^k}{2(2k-1)!}\left\{M^2-2(-1)^k N^{2k-1}(2k-1)!\sum_{i,j=0}^{N}\Delta_i^{2k}\Gamma_{ij}u\left(\frac{i}{N}\right)u\left(\frac{j}{N}\right)\right\}$$

$$\left\{F_x[F_\xi(H(x;\xi))]-\sum_{i=-k+1}^{N+k-1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^{2k}|Nx-i|^{2k-1})F\left[H\left(\frac{j}{N};\xi\right)\right]\right\} \tag{4.14}$$

where we again use the convention (4.8). If F is local in the sense that $F(u)$ only involves the values of u in the neighborhood of a point, the sums in the second term on the right involve at most $2k-1$ values of i and j. *Example: Let* $k=2$. Γ_{ij} is given by (3.22). We find that

$$\Delta^4|Nx-i|^3=\begin{cases}2\{4-6|Nx-i|^2+3|Nx-i|^3\}\\ 2\{2-|Nx-i|^3\}\\ 0\end{cases}$$

$$\begin{matrix}|Nx-i|\le 1\\ 1\le|Nx-i|\le 2\\ |Nx-i|\ge 2.\end{matrix} \tag{4.15}$$

Thus the interpolation function $g_j(x)$ is given by

$$g_j(x)=12(q+1-Nx)(Nx-q)[(q+2-Nx)\Gamma_{qj}$$

$$+(Nx+1-q)\Gamma_{q+1,\,j}]+(q+1-Nx)^3\delta_{qj})$$

$$+(Nx-q)^3\delta_{q+1,\,j}, \tag{4.16}$$

where the integer q is defined by

$$q<Nx\le q+1. \tag{4.17}$$

If $1<N\xi<N-1$, we let p be the integer such that

$$p+1<N\xi<p+2. \tag{4.18}$$

[6] Since Green's function is uniquely defined by (2.6), it does not depend upon the integer p used in the construction of H. The function Hdoes depend upon p.

Then we have

$$\frac{3N^3H(x;\xi)}{(N\xi-p)(N\xi-p-1)(p+2-N\xi)(p+3-N\xi)}=\begin{cases}0 & Nx\le p,\ Nx\ge p+3\\ \dfrac{(Nx-p)^3}{N\xi-p} & p\le Nx\le p+1\\ \dfrac{(Nx-p)^3}{N\xi-p}-\dfrac{3(Nx-p-1)^3}{N\xi-p-1} & p+1\le Nx\le N\xi\\ \dfrac{(p+3-Nx)^3}{p+3-N\xi}-\dfrac{3(p+2-Nx)^3}{p+2-N\xi} & N\xi\le Nx\le p+2\\ \dfrac{(p+3-Nx)^3}{p+3-N\xi} & p+2\le Nx\le p+3.\end{cases} \tag{4.19}$$

If $0<N\xi<1$, the definition (4.18) gives $p=-1$, so that the function H defined by (4.19) does not vanish at $x=0$. In this case we simply subtract the function

$$\frac{1}{2}(N\xi+1)^{-1}\Delta^4|Nx-1|^3 \tag{4.20}$$

from the right-hand side of (4.19) with $p=-1$ to obtain $H(x,\xi)$. Similarly, if $N-1<N\xi<N$, we subtract

$$\frac{1}{2}(N+1-N\xi)^{-1}\Delta^4|Nx-(N+1)|^3 \tag{4.21}$$

from the right-hand side of (4.19) with $p=N-2$.

Thus we can evaluate the bounds 4.14 explicitly. In the special case of linear interpolation we have

$$F(u)=u(\zeta). \tag{4.22}$$

If

$$1\le q<N\zeta<q+1\le N-1, \tag{4.23}$$

we find

$$F_x[F_\xi(H(x;\xi))]-\sum_{i=-1}^{N+1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^4|Nx-i|^3)F\left[H\left(\frac{j}{N};\xi\right)\right]$$

$$=N^{-3}(N\zeta-q)^2(q+1-N\zeta)^2\left\{4-\frac{\sqrt{3}}{3(1-y_1^{-2N})}\left[(q+2-N\zeta)^2(1-y_1^{-2q})(1-y_1^{-2(N-q)})\right.\right.$$

$$+2(q+2-N\zeta)(N\zeta-q+1)(1-y_1^{-2q})(1-y_1^{-2(N-q-1)})y_1^{-1}$$

$$\left.\left.+(N\zeta-q+1)^2(1-y_1^{-2(q+1)})(1-y_1^{-2(N-q-1)})\right]\right\}, \tag{4.24}$$

where

$$y_1=-2-\sqrt{3}.$$

The first factor on the right of (4.14) does not approach zero as $N\to\infty$ unless M^2 happens to be the exact value of the integral of $u^{[2]2}$. Thus the difference between the best upper and lower bounds in the linear interpolation problem with $N+1$ equally spaced points and with a given bound for $\int_0^1 u''^2dx$ is of the order $N^{-3/2}$. If a uniform bound for $|u''|$ is given, one can obtain bounds for $u(\zeta)$ which differ by a term of order N^{-2}. This shows that a bound for the square integral gives considerably less information than a bound for the maximum. The problem of finding best bounds when the maximum of $|u^{[2]}|$ is bounded is much more difficult than the problem treated here.

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Truncations in the Method of Intermediate Problems for Lower Bounds to Eigenvalues

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Two new procedures are developed for determining lower bounds to the eigenvalues of linear operators. The methods are based on the theory of semibounded self-adjoint operators in separable Hilbert space. Computation of the lower bounds is reduced to the solution of matrix problems. The procedures have immediate application in the estimation of eigenvalues and eigenvectors of differential operators appearing in quantum mechanics.

1. Introduction

In this paper we give two new procedures for determining lower bounds to eigenvalues of semibounded, self-adjoint operators. These procedures, while phrased in the language of Hilbert space, result in useful computational methods for the estimation of eigenvalues and eigenvectors of differential operators occurring in classical and quantum mechanics. Our procedures are based on the method of intermediate problems.

Intermediate problems, as introduced by A. Weinstein [7],[2] give lower bounds by changing the boundary conditions of differential operators. By relaxing the boundary conditions he obtains a solvable *base problem* that gives rough lower bounds to the eigenvalues. He then links the base problem to the given problem by a sequence of intermediate problems which can be solved in terms of the base problem and which improve the lower bounds.

Years ago H. Weyl [8] pointed out that rough lower bounds can also be obtained by changing the operator. This fact was used by N. Aronszajn [1] to construct intermediate problems for those cases in which a base problem can be obtained by neglecting a positive term in the operator. In the same paper Aronszajn systematized the method of intermediate problems for completely continuous operators in Hilbert space. As Aronszajn points out, the solution of the intermediate problems in his formulation requires the determination of the poles and zeros of a meromorphic function usually given as an infinite sum.

Recently H. F. Weinberger showed that it is possible to overcome the principal difficulties that arise in the intermediate problems of Weinsteint, Weinberger's result, which is a special case of his more general theory [6], gives lower bounds to the eigenvalues by the computation of the zeros of a polynomial.

In this paper we first sketch Aronszajn's method of intermediate problems for changes in the operator. Our presentation allows for the presence of continuous spectra and is in a form suitable for application to many quantum mechanical problems. In the second section we show how it is possible to modify the method so that lower bounds can be obtained by elementary calculations. These results give formulas parallel to some of those of Weinberger [6]. A feature of our procedure is that it can be used to calculate lower bounds to the energy levels of most atomic systems. The last section extends another method of one of the authors [2].

Part of the work presented here was started while the authors were in the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, and was summarized in [3].

2. Method of Intermediate Problems

Let us suppose that $\mathscr{H}$ is a separable Hilbert space with the inner product (u,v) and that A is a self-adjoint operator with domain $\mathscr{D}_A$ in $\mathscr{H}$. We will assume A is bounded below and that the first part of its spectrum is discrete. The eigenvalues and eigenvectors in this part of the spectrum will be denoted by λ_ν and $u_\nu (\nu=1, 2, \ldots)$ respectively; the first limit point of the spectrum will be called λ_∞. Thus we have

$$Au_\nu-\lambda_\nu u_\nu=0, \qquad (\nu=1,2,\ldots) \tag{1}$$

and

$$\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_\infty. \tag{2}$$

Suppose further that A can be written as the sum of two operators,

$$A=A^0+A', \tag{3}$$

where A^0 is a self-adjoint operator with a known spectral resolution E_λ^0 and A' is symmetric and positive. The domains of A^0 and A' are denoted by $\mathscr{D}_A^0$ and $\mathscr{D}_{A'}$, respectively, so that $\mathscr{D}_A=\mathscr{D}_{A^0}\cap\mathscr{D}_{A'}$. The positivity of A' is expressed by

$$(A'u,u)\geq 0 \tag{4}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

for every u in the domain of A'. Finally we assume that the first part of the spectrum of A^0 is discrete, and, listed in increasing order, the eigenvalues are

$$\lambda_1^0 \leq \lambda_2^0 \leq \ldots . \tag{5}$$

The corresponding eigenfunctions are $u_1^0, u_2^0, \ldots$.

Since $A^0 \leq A$, it follows[3] [4, p. 214] that

$$\lambda_\nu^0 \leq \lambda_\nu, \qquad (\nu = 1, 2, \ldots)$$

and

$$\lambda_\infty^0 \leq \lambda_\infty. \tag{6}$$

These inequalities go back to H. Weyl [8]. Thus the known eigenvalues of A^0 give rough lower bounds for the initial eigenvalues of A. For this reason the operator A^0 is called a *base operator* and the eigenvalue problem for A^0 is called a *base problem*.

The procedure of intermediate problems links the base operator A^0 to the *given operator* A by a sequence $\{A^k\}$ of *intermediate operators*. These operators satisfy the inequalities

$$A^0 \leq A^k \leq A^{k+1} \leq A, \qquad (k = 1, 2, \ldots); \tag{7}$$

and consequently their eigenvalues give *intermediate* lower bounds to those of A.

In order to define these intermediate operators, we introduce an auxiliary Hilbert space $\mathscr{H}'$. $\mathscr{H}'$ is just the completion of $\mathscr{D}_{A'}$ in the norm generated by the inner product $[u,v]$ defined by

$$[u,v] = (A'u, v). \tag{8}$$

For simplicity we will assume that A' is positive definite; the indefinite case requires only a slight extension of the theory. Let $\{p_i\}$ be a linearly independent set of vectors in $\mathscr{D}_{A'}$; note that since A' is positive definite the sequence $\{A'p_i\}$ is also linearly independent. Define P^k to be the projection in $\mathscr{H}'$ on the span $\mathscr{S}^k$ of the first k elements of $\{p_i\}$. For an arbitrary vector v in $\mathscr{H}'$, $P^k v$ is given by

$$P^k v = \sum_{i,j=1}^{k} [v, p_i] b_{ij} p_j, \tag{9}$$

where the constants b_{ij} are the elements of the matrix inverse to that with the elements $(A'p_i, p_j)$. Since the subspaces $\mathscr{S}^k$ are increasing with k; i.e.,

$$\mathscr{S}^k \subset \mathscr{S}^{k+1} \subset \mathscr{H}', \qquad (k = 1, 2, \ldots), \tag{10}$$

the corresponding projections satisfy the inequalities,

$$0 \leq P^k \leq P^{k+1} \leq I, \tag{11}$$

where I is the identity operator in $\mathscr{H}'$. We now form the operator $A'P^k$, which is well defined on $\mathscr{D}_A'$. For an arbitrary vector w in $\mathscr{D}_{A'}$, $A'P^k w$ is given by

$$A'P^k w = \sum_{i,j=1}^{k} (w, A'p_i) b_{ij} A'p_j, \quad (k = 1, 2, \ldots). \tag{12}$$

In the form given on the right in eq (12) $A'P^k$ has an obvious extension to all of $\mathscr{H}$, and there it is a symmetric completely continuous operator. Henceforward we will regard $A'P^k$ as thus extended.

It follows from eq (11) that in $\mathscr{H}$ the operators $A'P^k$ satisfy the inequalities,

$$0 \leq A'P^k \leq A'P^{k+1} \leq A', \qquad (k = 1, 2, \ldots). \tag{13}$$

The intermediate operators A^k are now defined by

$$A^k = A^0 + A'P^k, \qquad (k = 1, 2, \ldots); \tag{14}$$

k is called the *order* of the intermediate operator. Since the operators $A'P^k$ are symmetric and bounded, it follows [5, p. 301] that the operators A^k are self-adjoint and have the same domain as A^0. And moreover, since $A'P^k$ is for each k a completely continuous operator, each of the operators A^k has exactly the same limit points in its spectrum as does A^0 [5, p. 367]. From eq (13) it follows directly that

$$A^0 \leq A^k \leq A^{k+1} \leq A, \qquad (k = 1, 2, \ldots).$$

These inequalities are just those stated in (7), and they show that the operators A^k defined by eq (14) are in fact intermediate operators. This means that in the initial discrete part of the spectra the eigenvalues satisfy the inequalities

$$\lambda_\nu^0 \leq \lambda_\nu^k \leq \lambda_\nu^{k+1} \leq \lambda_\nu, \qquad (\nu = 1, 2, \ldots; \infty), \tag{15}$$

where the superscripts on the eigenvalues indicate the order of the intermediate operator to which they belong.

We now turn to the problem of determining the eigenvalues and eigenvectors of the intermediate operators A^k. We will confine our attention to those eigenvalues which lie below the first limit point λ_∞^0 common to all operators A^k.

The eigenvalue equation for A^k,

$$A^k u - \lambda u = 0, \tag{16}$$

can be written in a more useful form by using the defining eq (14) for A^k. This gives

$$A^0 u - \lambda u = -A'P^k u; \tag{17}$$

and by using eq (12) we obtain

$$A^0 u - \lambda u = -\sum_{i,j=1}^{k} (u, A'p_i) b_{ij} A'p_j. \tag{18}$$

For convenience we designate the quantities $\sum_{i=1}^{k} (u, A'p_i) b_{ij}$ by α_j, so that the equations to be satis-

[3] $A^0 \leq A$ means $\mathscr{D}_{A^0} \supset \mathscr{D}_A$ and $(A^0 u, u) \leq (Au, u)$ for every u in $\mathscr{D}_A$.

fied become

$$A^0u-\lambda u=-\sum_{j=1}^{k}\alpha_jA'p_j, \tag{19}$$

and

$$\sum_{j=1}^{k}\alpha_j(p_j,A'p_i)=(u,A'p_i), \qquad (i=1,2,\ldots,k). \tag{20}$$

If u is to be an eigenvector corresponding to a value λ which is not in the spectrum of A^0, then u may be written

$$u=-\sum_{j=1}^{k}\alpha_jR_\lambda A'p_j, \tag{21}$$

where R_γ is the resolvent operator for A^0.

The eqs (20) become

$$\sum_{j=1}^{k}\alpha_j(p_j+R_\lambda A'p_j,A'p_i)=0, \qquad (i=1,2,\ldots,k). \tag{22}$$

All of the α's cannot vanish if u is to be nontrivial; therefore λ must be such that the determinental equation

$$|(p_j+R_\lambda A'p_j,A'p_i)|=0 \tag{23}$$

is satisfied.

Equations (22) have solutions for each value λ that is not in the spectrum of A^0 and that satisfies eq (23). The number of solutions corresponding to each such root is just the nullity of the coefficient matrix and, according to eq (21), these solutions give the eigenvectors of A^k associated with that root. These eigenvectors are clearly linearly independent since R_λ is one-to-one on $\mathscr{H}$.

For each value λ that is an eigenvalue of the operator A^0, say λ_ν^0, the procedure is somewhat different. Let $\mathscr{M}_{\lambda_\nu^0}$ be the characteristic subspace of A^0 at the eigenvalue λ_ν^0. Suppose that it is spanned by the eigenvectors $\{v_1, v_2, \ldots, v_l\}$. It is clear that if u is an eigenvector of A^k corresponding to λ_ν^0, then in addition to the eq (19) and (20) the further restrictions,

$$\sum_{j=1}^{k}\alpha_j(A'p_j,v_{i-k})=0, \quad (i=k+1,k+2,\ldots,k+l) \tag{24}$$

must be satisfied. If we assume that eq (24) are satisfied, u is given by

$$u=-\sum_{j=1}^{k}\alpha_jR'_{\lambda_\nu^0}A'p_j-\sum_{j=k+1}^{k+l}\alpha_jv_{j-k}, \tag{25}$$

where by $R'_{\lambda_\nu^0}$ we mean the restriction of the resolvent operator for A^0 to the part of our Hilbert space orthogonal to $\mathscr{M}_{\lambda_\nu^0}$. The eqs (20) become

$$\sum_{j=1}^{k}\alpha_j(p_j+R'_{\lambda_\nu^0}A'p_j,\ A'p_i)+\sum_{j=k+1}^{k+l}\alpha_j(v_{j-k},A'p_i)=0,$$
$$(i=1,2,\ldots,k). \tag{26}$$

These, together with eq (24), determine the constants α_i. The necessary and sufficient condition for the existence of nontrivial solutions is the vanishing of the determinant of the coefficients; i.e.,

$$\det\begin{pmatrix}(p_j+R'_{\lambda_\nu^0}A'p_j,A'p_i) & (v_{j-k},A'p_i)\\ \cdots\cdots & \cdots\cdots\\ (A'p_j,v_{i-k}) & 0\end{pmatrix}=0. \tag{27}$$

Let us assume that (27) holds; that is, that λ_ν^0 is an eigenvalue of A^k as well as of A^0. The eigenvectors of A^k corresponding to λ_ν^0 are then determined in the following way: The eqs (24) and (26) have linearly independent solutions $\{\alpha_1^\sigma, \alpha_2^\sigma, \ldots, \alpha_{k+l}^\sigma\}$, $(\sigma=1, 2, \ldots, n)$, where n is the nullity of the coefficient matrix displayed in (27). By (25) these give rise to n vectors u^σ that are eigenvectors of A^k for the value λ_ν^0. These are also linearly independent, for if C_σ are constants such at $\Sigma_{\sigma=1}^{n}C_\sigma u^\sigma=0$, then by the expression (25) for u^σ we must have

$$R'_{\lambda_\nu^0}\Big(\sum_{\sigma=1}^{n}C_\sigma\sum_{j=1}^{k}\alpha_j^\sigma A'p_j\Big)+\sum_{\sigma=1}^{n}C_\sigma\sum_{j=k+1}^{k+l}\alpha_j^\sigma v_{j-k}=0. \tag{28}$$

But since $R'_{\lambda_\nu^0}$ is a one-to-one operator on the orthogonal complement of $\mathscr{M}_{\lambda_\nu^0}$, eq (28) implies the separate statements,

$$\sum_{j=1}^{k}\Big(\sum_{\sigma=1}^{n}C_\sigma\alpha_j^\sigma\Big)A'p_j=0, \qquad \sum_{j=k+1}^{k+l}\Big(\sum_{\sigma=1}^{n}C_\sigma\alpha_j^\sigma\Big)v_{j-k}=0. \tag{29}$$

The linear independence of the vectors $A'p_i$ and of the vectors v_j imply that

$$\sum_{\sigma=1}^{n}C_\sigma\alpha_j^\sigma=0, \qquad (j=1,2,\cdots,k+l).$$

But since the vectors $\{\alpha_1^\sigma, \alpha_2^\sigma, \ldots, \alpha_{k+l}^\sigma\}$ are linearly independent, each of the constants C_σ must vanish, and thus the vectors u^σ are linearly independent.

As we observed previously, the part of the spectrum of A^k that lies below λ_ω^0 is discrete. The eigenvalues there are completely determined from eqs (23) and (27), and the eigenvectors are found from corresponding solutions of the linear equations. These eigenvalues give intermediate lower bounds to those in the first part of the spectrum of A.

For the sake of completeness and later use, we sketch a proof of convergence of the eigenvalues of the intermediate problems. The conditions that we use are those stated by Aronszajn [1]. We assume that A^0 and A have completely continuous inverses, that A' is bounded relative to A^0, and that the set $\{p_i\}$ is complete in $\mathscr{H}'$. For convenience we will assume that A^0 is positive definite; the modifications for A^0 bounded below are straightforward.

The purpose of the proof is to show that the inverse operators converge uniformly to A^{-1}. From this the convergence of the eigenvalues follows immediately [5, p. 372].

Since the inverse operators $(A^k)^{-1}$ form a decreasing sequence of symmetric operators bounded below by A^{-1}, the sequence converges strongly to some symmetric operator [5, p. 263]. The strong convergence to A^{-1} is guaranteed by the inequalities that hold for any t in $\mathscr{H}$,

$$\|[(A^k)^{-1}-A^{-1}]t\|^4 \leq \|(A^0)^{-1}-A^{-1}\|^3\|t\|^2([(A^k)^{-1}-A^{-1}]t,t), \quad (30)$$

and

$$([(A^k)^{-1}-A^{-1}]t,t)^2 \leq [(I-P^k)A^{-1}t,(I-P^k)A^{-1}t] \cdot (A'(A^k)^{-1}t,(A^k)^{-1}t). \quad (31)$$

The first of these inequalities follows by using the Schwarz inequality in the form generated by $(A^k)^{-1}-A^{-1}$ and then using the fact that $(A^k)^{-1}\leq(A^0)^{-1}$. The second is obtained by writing $(A^k)^{-1}-A^{-1}=(A^k)^{-1}A'(I-P^k)A^{-1}$, and then using the Schwarz inequality in the form generated by A'. The first factor on the right in (31) vanishes with increasing k since the set $\{p_i\}$ is complete in $\mathscr{H}'$. The second remains bounded since A' is bounded relative to A^0. The strong convergence of $(A^k)^{-1}$ to A^{-1} is thus insured. However, since each operator $(A^k)^{-1}$ as well as A^{-1} is completely continuous and symmetric, the monotonic strong convergence implies uniform convergency by an easily proved analog of Dini's theorem.

REMARK: We observe that if the span of the vectors $\{p_i\}_{i=1}^k$ contains some eigenvectors $\{u_{\nu_1}, u_{\nu_2} \ldots, u_{\nu_j}\}$ of A, $(j\leq k)$, then $A^k=A$ on the span of these u's so that $\lambda_{\nu_1}, \lambda_{\nu_2}, \ldots, \lambda_{\nu_j}$ are eigenvalues of A^k as well as A, and they have the corresponding eigenvectors $u_{\nu_1}, u_{\nu_2}, \ldots, u_{\nu_j}$, respectively. Investigations based on this property lead to statements of the following type:

THEOREM: Let A be an operator with a purely point spectrum and let λ_ν be an eigenvalue of A that is less than λ_∞^0, then there exists an intermediate problem (of sufficiently high order) which has $\lambda_\mu^k=\lambda_\mu$ and $u_\mu^k=u_\mu$ for $\mu=1,2, \ldots, \nu$.

3. Truncation of the Base Operator

The principal difficulties in the computation of approximating eigenvalues and eigenvectors by the method just described arise from the fact that the resolvent operator R_λ for the base problem rarely is known in closed form. In most cases it can be expressed only by use of infinite sums or integrals; i.e., in our cases by

$$R_\lambda w=\sum_{\nu=1}\frac{(w,u_\nu^0)u_\nu^0}{\lambda_\nu^0-\lambda}+\int_{\lambda_\infty^0-0}^{\infty}\frac{dE_\mu^0 w}{\mu-\lambda}.$$

We overcome this difficulty by changing the base problem so that it has a simple resolvent and still provides lower bounds by the method of intermediate problems. These bounds are computed from equations identical with those given earlier except that the resolvent is expressed by a finite sum with a remainder in closed form.

Let us assume that A^0 has infinitely many eigenvalues less than λ_∞ (the modifications for the cases in which there are only a finite number will be immediate). Define the *truncated operators* $A^{i,0}$ by the equations

$$A^{i,0}=A^0E_{\lambda_i^0}^0+\lambda_{i+1}^0(I-E_{\lambda_i^0}^0), \qquad (i=1,2,\ldots), \quad (32)$$

and denote the corresponding resolutions of the identity by $E_\lambda^{i,0}$. It is clear that each operator $A^{i,0}$ is bounded and symmetric, has the eigenvalues and eigenvectors λ_ν^0, u_ν^0 for $\nu=1,2, \ldots, i$, and also has λ_{i+1}^0 as an eigenvalue of infinite multiplicity. The resolutions of the identity may be written down immediately. They are

$$E_\lambda^{i,0}=\begin{Bmatrix} E_\lambda^0, & \lambda<\lambda_{i+1}^0 \\ I, & \lambda_{i+1}^0\leq\lambda \end{Bmatrix}, \qquad (i=1,2,\ldots). \quad (33)$$

An operator $A^{i,0}$ is called a *truncation* of A^0 of order i. These operators satisfy the fundamental inequalities

$$A^{i,0}\leq A^{i+1,0}\leq A^0, \qquad (i=1,2,\ldots), \quad (34)$$

since from the spectral theorem it follows that

$$([A^{i+1,0}-A^{i,0}]u,u)=(\lambda_{i+2}^0-\lambda_{i+1}^0)\int_{\lambda_{i+2}^0-0}^{\infty}d(E_\lambda u,u)\geq 0.$$

for each u in $\mathscr{H}$, and that

$$([A^0-A^{i,0}]u,u)=\int_{\lambda_{i+1}^0-0}^{\infty}(\lambda-\lambda_{i+1}^0)d(E_\lambda^0 u,u)\geq 0$$

for each u in $\mathscr{D}_{A^0}$.

The new intermediate operators $A^{i,k}$ based on $A^{i,0}$ are defined by

$$A^{i,k}=A^{i,0}+A'P^k, \qquad (i,k=1,2,\ldots). \quad (36)$$

These are bounded, symmetric, and are monotonically increasing in i and k; i.e.,

$$A^{i,k}\leq A^{i+1,k}\leq A^k\leq A, \qquad (i,k=1,2,\ldots), \quad (37)$$

and

$$A^{i,k}\leq A^{i,k+1}\leq A^{i,0}+A'\leq A, \qquad (i,k=1,2,\ldots). \quad (38)$$

The montonicity in the first index follows from (34) and in the second index from (13). The inequalities (37) and (38) imply the parallel inequalities for the corresponding eigenvalues,

$$\left.\begin{aligned} &\lambda_\nu^{i,k}\leq\lambda_\nu^{i+1,k}\leq\lambda_\nu^k\leq\lambda_\nu \\ &\lambda_\nu^{i,k}\leq\lambda_\nu^{i,k+1}\leq\lambda_\nu \end{aligned}\right\}, \qquad (\nu=1,2,\ldots), \quad (39)$$

where by $\lambda_\nu^{i,k}$ we mean the νth eigenvalue of $A^{i,k}$.

Since $A^{i,k}$ and $A^{i,0}$ differ by a completely continuous operator, the limit point λ_{i+1}^0 persists and is the only limit point of the spectrum of the intermediate operator $A^{i,k}$ as the order k of the operator is increased [5, p. 367]. Further, the inequalities (39) show that for any k there are at most i eigenvalues of $A^{i,k}$ less than λ_{i+1}^0.

The spectrum of each operator $A^{i,k}$ can be determined easily by the usual procedure for intermediate problems. The eigenvalue equation for $A^{i,k}$ is stated by

$$A^{i,k}u-\lambda u=0,$$

or equivalently (in the notation of the preceding section)

$$A^{i,0}u-\lambda u=-\sum_{j=1}^{k}\alpha_j A'p_j,$$

$$\sum_{j=1}^{k}\alpha_j(p_j,A'p_l)=(u,A'p_l), \qquad (l=1,2,\ldots,k).$$

If u is an eigenvector corresponding to a value λ that is not in the spectrum of $A^{i,0}$, then u may be written

$$u=-\sum_{j=1}^{k}\alpha_j R_\lambda^i A'p_j, \tag{40}$$

where R_λ^i is the resolvent operator of $A^{i,0}$. We note here that for any vector v in $\mathcal{H}$, $R_\lambda^i v$ is given by the closed expression,

$$R_\lambda^i v=\sum_{\nu=1}^{i}\frac{(v,u_\nu^0)\,u_\nu^0}{\lambda_\nu^0-\lambda}+\frac{1}{\lambda_{i+1}^0-\lambda}\left[v-\sum_{\nu=1}^{i}(v,u_\nu^0)\,u_\nu^0\right]. \tag{41}$$

The eqs (22) become

$$\sum_{j=1}^{k}\alpha_j(p_j+R_\lambda^i A'p_j,A'p_l)=0, \qquad (l=1,2,\cdots,k). \tag{42}$$

As a consequence, the necessary and sufficient condition for such a λ to be an eigenvalue is that it be a root of the equation

$$|(p_j+R_\lambda^i A'p_j,A'p_l)|=0. \tag{43}$$

The multiplicity of each such eigenvalue is the nullity of the coefficient matrix of eq (42). The eigenvectors of $A^{i,k}$ are determined from the solutions of eq (42) by eq (40).

Now suppose that λ_ν^0 is an eigenvalue of $A^{i,0}$ that is less than λ_{i+1}^0 and that the characteristic subspace $\mathcal{M}_{\lambda_\nu}^0$ of $A^{i,0}$ corresponding to λ_ν^0 is spanned by $\{v_1, v_2, \ldots, v_l\}$. If u is to be an eigenvector of $A^{i,k}$ corresponding to λ_ν^0, then u is given by the equation

$$u=-\sum_{j=1}^{k}\alpha_j R_{\lambda_\nu^0}^{i'}A'p_j-\sum_{j=k+1}^{k+l}\alpha_j v_{j-k}, \tag{44}$$

where the constants α_j must satisfy the equations

$$\sum_{j=1}^{k}\alpha_j(p_j+R_{\lambda_\nu^0}^{i'}A'p_j,A'p_r)+\sum_{j=k+1}^{k+l}\alpha_j(v_{j-k},A'p_r)=0, \qquad (r=1,2,\ldots,k) \tag{45}$$

and

$$\sum_{j=1}^{k}\alpha_j(A'p_j,v_{r-k})=0, \quad (r=k+1,k+2,\ldots,k+l). \tag{46}$$

The linear independence of the eigenvectors arising from linearly independent sets of α's may be demonstrated as in the preceding section.

The multiplicity of λ_ν^0 as an eigenvalue of $A^{i,k}$ is thus just the nullity of the matrix

$$\begin{pmatrix}(p_j+R_{\lambda_\nu^0}^{i'}A'p_j,A'p_r) & (v_{j-k},A'p_r)\\ \hline (A'p_j,v_{r-k}) & 0\end{pmatrix}. \tag{47}$$

We have found all of the eigenvalues of $A^{i,k}$ that are not equal to λ_{i+1}^0. The complete continuity of $A'P^k$ implies that λ_{i+1}^0, the only limit point of the spectrum of $A^{i,0}$, remains as the only limit point of the spectrum of $A^{i,k}$. Thus the total spectrum of $A^{i,k}$ is determined. The spectral subspace of $A^{i,k}$ associated with λ_{i+1}^0 is just the orthogonal complement of the span of all the eigenvectors already determined for the values of λ not equal to λ_{i+1}^0.

Thus, as stated at the beginning of this section, the method of truncation justifies the replacement of the base problem resolvent by a suitable approximation in closed form. The necessary computations can be carried out by routine techniques.

The convergence of the eigenvalues of the truncated intermediate operator $A^{i,k}$ to those of A^k can be shown under the assumption that A^0 and A have completely continuous inverses. We demonstrate the uniform convergence of $(A^{i,k})^{-1}$ to $(A^k)^{-1}$ for each fixed k. For convenience we will assume that A° is positive definite. First we note that the operators $(A^{i,0})^{-1}$ converge in norm to $(A^0)^{-1}$ as i becomes infinite. This follows directly from the spectral theorem, for we have

$$(A^0)^{-1}=\int^{\infty}\lambda^{-1}dE_\lambda^0,$$

and

$$(A^{i,0})^{-1}=\int^{\lambda_{i+1}^0-0}\lambda^{-1}dE_\lambda^0+\frac{1}{\lambda_{i+1}^0}\int_{\lambda_{i+1}^0-0}^{\infty}dE_\lambda^0,$$

so that

$$(A^0)^{-1}-(A^{i,0})^{-1}=\int_{\lambda^0_{i+1}-0}^{\infty}\left(\frac{1}{\lambda}-\frac{1}{\lambda^0_{i+1}}\right)dE^0_\lambda.$$

Consequently we have

$$\|(A^0)^{-1}-(A^{i,0})^{-1}\|\leq\frac{1}{\lambda^0_{i+1}}. \tag{48}$$

Since λ^0_{i+1} becomes infinite with i, the norm convergence of $(A^{i,0})^{-1}$ is clear. This implies that $(A^{i,k})^{-1}$ converges in norm to $(A^k)^{-1}$. The proof goes as follows. We start from the simple identity

$$A^{i,k}-A^k=A^{i,0}-A^0.$$

From this it follows immediately that

$$A^{i,k}[(A^k)^{-1}-(A^{i,k})^{-1}]A^k=A^{i,0}[(A^0)^{-1}-(A^{i,0})^{-1}]A^0,$$

and thus

$$(A^k)^{-1}-(A^{i,k})^{-1}=(A^{i,k})^{-1}A^{i,0}\\ [(A^0)^{-1}-(A^{i,0})^{-1}]A^0(A^k)^{-1}. \tag{49}$$

By writing $(A^{i,k})^{-1}A^{i,0}$ as $I-(A^{i,k})^{-1}A'P^k$ it is evident that $(A^{i,k})^{-1}A^{i,0}$ is an operator that can be bounded independently of i since $\|(A^{i,k})^{-1}\|\leq\frac{1}{\lambda^0_1}$. Since $A^0(A^k)^{-1}$ is also bounded for each fixed k, the convergence in norm of $(A^{i,k})^{-1}$ to $(A^k)^{-1}$ follows directly from (48) and (49).

The uniform convergence of $(A^{i,k})^{-1}$ to $(A^k)^{-1}$ and that of $(A^k)^{-1}$ to A^{-1} demonstrated in section 2 are sufficient to insure that the eigenvalues of $A^{i,k}$ converge to those of A as i and k become large.

4. Truncation by Choice of Elements

In this section we discuss an extension of another procedure [2] for truncating the resolvent R_λ that appears in section 2. This can be accomplished by choosing the elements p_i in suitable ways instead of changing the base problem operator as has been done in previous sections.

We assume that the vectors $\{p_1, p_2, \ldots, p_k\}$ which determine A^k have been chosen so that

$$A'p_i=\sum_{\nu=1}^{N}\beta_{i\nu}u^0_\nu, \qquad (i=1, 2, \ldots, k). \tag{50}$$

That is, it is assumed that each of quantities $A'p_i(i=1, 2, \ldots, k)$ can be expressed in terms of a finite number N of eigenvectors of A^0. The number N will of course depend on k. The quantities $(R_\lambda A'p_i, A'p_j)$ which appear in section 2 will have the simple expressions

$$(R_\lambda A'p_i, A'p_j)=\sum_{\nu=1}^{N}\frac{\beta_{i\nu}\bar{\beta}_{\nu j}}{\lambda^0_\nu-\lambda}, \tag{51}$$

and thus are effectively truncated by the choice of elements (50). The computations for the eigenvalues and eigenvectors may be carried out directly from the equations of section 2.

In the present case, however, it is simpler to proceed directly. We have already shown that the expression A^ku can be written

$$A^ku=A^0u+\sum_{i,j=1}^{k}(u, A'p_i)b_{ij}A'p_j,$$

where, as before, the quantities b_{ij} are the elements of the matrix inverse to that having $(A'p_i, p_j)$ as elements. Using (50) this expression becomes

$$A^ku=A^0u+\sum_{i,j=1}^{k}\sum_{\mu,\nu=1}^{N}(u, u^0_\nu)\bar{\beta}_{i\nu}b_{ij}\beta_{j\mu}u^0_\mu. \tag{52}$$

We can now obtain all of the eigenvalues and eigenvectors of A^k by inspection. First consider those vectors in $\mathscr{D}_{A^0}$ that are orthogonal to every eigenvector of A^0. If u is such a vector, then (52) shows that $A^ku-\lambda u=A^0u-\lambda u$. Clearly no such a vector can be an eigenvector of A^k. Further, if u is an eigenvector u^0_σ of A^0 with an eigenvalue λ^0_σ and is not used in the sums in (50), then by (52) we have $A^ku^0_\sigma-\lambda^0_\sigma u^0_\sigma=A^0u^0_\sigma-\lambda^0_\sigma u^0_\sigma=0$. Hence each such vector is an eigenvector of A^k with the same eigenvalue as for A^0_σ.

It remains to consider those vectors u which are linear combinations of the vectors $u^0_\nu(\nu=1, 2, \ldots, N)$ used in the sums in (50). There are exactly N linearly independent vectors of this kind that are eigenvectors of A^k. These are determined as follows. We write u as

$$u=\sum_{\nu=1}^{N}\gamma_\nu u^0_\nu,$$

and insert this in the eigenvalue equation $A^ku-\lambda u=0$, where A^k is given by (52). The linear independence of the vectors u^0_ν leads to the equivalent algebraic system,

$$\sum_{\nu=1}^{N}\gamma_\nu\left(\sum_{i,j=1}^{k}\bar{\beta}_{i\nu}b_{ij}\beta_{j\mu}+(\lambda^0_\nu-\lambda)\delta_{\nu\mu}\right)=0,\\ (\mu=1,2, \ldots, N). \tag{53}$$

Consequently the determination of the intermediate eigenvalues and eigenvectors depends only on the solution of a matrix eigenvalue problem.

This procedure can be especially useful when the operator A' is a multiplicative operator, as for example in the spheriodal wave equation.[4] In the prolate case the eigenvalue equation is stated by

$$Au-\lambda u=0,$$

[4] The numerical applications in this and other cases are presently being carried out by W. Börsch-Supan.

in which A is the essentially self-adjoint operator given by

$$Au=-\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]+\frac{m^2}{1-x^2}u+c^2x^2u$$

on the set of all functions continuous on $[-1, +1]$ that are twice continuously differentiable on $(-1, +1)$ and for which Au belongs to $\mathscr{L}^2(-1, +1)$. Here m is a non-negative integer and c is a real constant. A base operator is obtained by omitting the positive term $A'=c^2x^2$. The solutions of the base problem are associated Legendre functions. The choice $p_l=u_l^0=P_{m+l-1}^m$ and the recursion relation,

$$xP_n^m=\frac{(n-m+1)P_{n+1}^m+(n+m)P_{n-1}^m}{2n+1},$$

allow one to express $A'p_l$ as a linear combination of u_{l-2}^0, u_{l-1}^0, u_l^0, u_{l+1}^0, and u_{l+2}^0. Other examples of this sort can be found easily.

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Comparison Theorems for Symmetric Functions of Characteristic Roots[1]

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Several theorems are proved that give necessary and sufficient conditions for $A-B$ to be positive semidefinite Hermitian. The conditions are in terms of the comparison of elementary symmetric functions of the characteristic roots of $A+X$ and $B+X$ as X varies over positive definite Hermitian matrices.

1. Introduction

In a recent paper [4][2] M. Stone obtained the following result in reproving certain theorems of Ehrenfeld [1]: *if F and G are positive definite n-square Hermitian matrices, then*

$$d(I+AF)\geq d(I+AG) \tag{1}$$

for all positive definite n-square Hermitian matrices A if and only if $F-G$ is positive semidefinite Hermitian. Here d denotes determinant and henceforth $A>0$ (≥ 0) will signify that A is positive (positive semidefinite) Hermitian. $F\geq G$ ($F>G$) will mean $F-G\geq 0$ ($F-G>0$). Note that since $A>0$ if and only if $A^{-1}>0$ the condition (1) is the same as saying

$$d(A+F)\geq d(A+G) \tag{2}$$

for all $A>0$. It is in the form (2) that we investigate what happens when we replace d by some other elementary symmetric function of the characteristic roots. To be specific, suppose $E_r(x_1, \ldots, x_n)$ denotes the r^{th} elementary symmetric function of the indicated variables:

$$E_r(x_1, \cdots, x_n)=\sum_{1\leq i_1<\ldots<i_r\leq n}\prod_{t=1}^{r} x_{i_t}.$$

For a fixed r, $1\leq r\leq n$, let $E_r(A)$ denote $E_r(\lambda_1(A), \ldots, \lambda_n(A))$ where $\lambda_i(A)$, $i=1, \ldots, n$, are the characteristic roots of A. If the $\lambda_i(A)$ are real we will choose our notation so that $\lambda_1(A)\geq \ldots \geq \lambda_n(A)$. Then the problem we pose is to find conditions on the characteristic roots of F and G such that

$$E_r(A+F)\geq E_r(A+G) \text{ for all } A\geq 0.$$

We have

THEOREM 1. *Assume $F\geq 0$, $G\geq 0$, and $1\leq r\leq n$. If*

$$E_{r-p}(\lambda_n(F), \ldots, \lambda_{p+1}(F))\geq E_{r-p}(\lambda_1(G), \ldots, \lambda_{n-p}(G)), \tag{3}$$

$p=0, \ldots, r-1$

it follows that

$$E_r(A+F)\geq E_r(A+G) \text{ for all } A\geq 0. \tag{4}$$

In case $G=I$, (3) becomes

$$E_{r-p}(\lambda_n(F), \cdots, \lambda_{p+1}(F))\geq \binom{n-p}{r-p}, \quad p=0, \cdots, r-1, \tag{5}$$

which is both necessary and sufficient for

$$E_r(A+F)\geq E_r(A+I) \quad \text{for all } A\geq 0. \tag{6}$$

We remark that for $r=n$ the second part of theorem 1 simply becomes: $\lambda_n(F)\geq 1$ *if and only if* $d(A+F)\geq d(A+I)$ *for all* $A\geq 0$. Now in (1) multiply both sides by $d(A^{-1}R^*R)$ where R is a nonsingular matrix satisfying $R^*GR=I$, $R^*FR=K$ and we find that

$$d(R^*A^{-1}R+K)\geq d(R^*A^{-1}R+I).$$

Now as A runs over all positive definite matrices so does $R^*A^{-1}R$ and from the above remark we conclude that $\lambda_n(K)\geq 1$. The characteristic roots of K are just the characeristic roots of $G^{-1}F$ and thus

$$G^{-1/2}FG^{-1/2}\geq I, \qquad F\geq G,$$

and the result in [4] follows. A somewhat stronger multiplicative analogue of theorem 1 is available.

THEOREM 2. *Assume $F\geq 0$, $G\geq 0$, and $1\leq r\leq n$. Then*

$$E_r(AF)\geq E_r(AG) \text{ for all } A\geq 0 \tag{7}$$

[1] This research was supported in part by the Office of Naval Research.
[2] Figures in brackets indicate the literature references at the end for this paper.

if and only if

$$\prod_{j=1}^{r} \lambda_j(F^{-1}G) \leq 1. \tag{8}$$

In order to prove these results we use some results concerning compound matrices and Grassmann products [2]. We give the coordinate definition of these items and list several of their properties. If $x_1, \ldots, x_r$ are vectors in the unitary space of n-tuples, $x_i=(x_{i1}, \ldots, x_{in})$, $1 \leq r \leq n$, then $x_1 \wedge \ldots \wedge x_r$ is the $\binom{n}{r}$-tuple whose coordinates are the r-square subdeterminants of the $r \times n$ matrix (x_{ij}), $i=1, \ldots, r$, $j=1, \ldots, n$ arranged in lexicographic order. If A is an n-square matrix, then $C_r(A)$ is the $\binom{n}{r}$-square matrix whose entries are the r-square subdeterminants of A arranged in doubly lexicographic order according to the row and column indices of A. That is, if $1 \leq i_1 < \ldots < i_r \leq n$ and $1 \leq j_1 < \ldots < j_r \leq n$ are two increasing sets of r integers then the $(i_1, \ldots, i_r)$, $(j_1, \ldots, j_r)$ element of $C_r(A)$ is the determinant of the matrix $A[i_1, \ldots, i_r | j_1, \ldots, j_r]$ whose (s, t) entry is $a_{i_s j_t}$. If $i_s=j_s$, $s=1, \ldots, r$ we denote the corresponding principal submatrix by $A[i_1, \ldots, i_r]$. If $A \geq 0$ then $C_r(A) \geq 0$; $C_r(A)x_1 \wedge \ldots \wedge x_r = Ax \wedge \ldots \wedge Ax_r$; if $(\ ,\ )$ is the usual unitary inner product in the space of m-tuples then $(x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots \wedge y_r) = d\{(x_i, y_j)\}$, $i, j=1, \ldots, r$.

We remark that the condition (3) of theorem 1 will not imply that $F \geq G$ in the case $r < n$. For even if $G=I$, we can easily construct F so that $\lambda_n(F) < 1$ and yet (3) holds for $p=0, \ldots, r-1$. However, there is a direct generalization of Stone's result to the compound matrix.

THEOREM 3. *If $F>0$, $G>0$ and $1 \leq r \leq n$, then*

$$C_r(I+AF) \geq C_r(I+AG) \text{ for all } A>0$$

if and only if $F \geq G$. (9)

Once again, $r=n$ is precisely the result in [4].

2. Proofs

We prove theorem 1. Let $A=UXU^*$, U unitary, $X=\operatorname{diag}(x_1, \ldots, x_n)$. Then $E_r(A+F)-E_r(A+G) = E_r(U^*AU+U^*FU)-E_r(U^*AU+U^*GU) = E_r(X+K)-E_r(X+H)$ where $K=U^*FU \geq 0$, $H=U^*GU \geq 0$. Let

$$\varphi(x_1, \ldots, x_n: U) = E_r(X+K)-E_r(X+H).$$

Then

$$\varphi(x_1, \ldots, x_n: U) = \sum_{1 \leq \alpha_1 < \ldots < \alpha_r \leq n} \{d[\operatorname{diag}(x_{\alpha_1}, \ldots, x_{\alpha_r}) + K[\alpha_1, \ldots, \alpha_r]] - d[\operatorname{diag}(x_{\alpha_1}, \ldots, x_{\alpha_r}) + H[\alpha_1, \ldots, \alpha_r]]\}. \tag{10}$$

We see from (10) that φ is a polynomial in $x_1, \ldots, x_n$ of degree r in which the nonconstant terms form a multilinear polynomial. Let

$$\varphi_{i_1, \ldots, i_p}(x_1, \ldots, x_n: U) = \frac{\partial^p \varphi}{\partial x_{i_1} \ldots \partial x_{i_p}}(x_1, \ldots, x_n: U)$$

and

$$\varphi^{\circ}_{i_1 \ldots i_p} = \varphi_{i_1 \ldots i_p}(0, \ldots, 0: U).$$

Then

$$\varphi(x_1, \ldots, x_n: U) = \sum_{p=0}^{r} \sum_{1 \leq i_1 < \ldots < i_p \leq n} x_{i_1} \ldots x_{i_p} \varphi^{\circ}_{i_1 \ldots i_p}, \tag{11}$$

where the term corresponding to $p=0$ is just $\varphi(0, \ldots, 0: U)$. Let w denote the increasing sequence $i_1 < \ldots < i_p$. Unless w is a subset of $\alpha_1 < \ldots < \alpha_r$ the partial derivative satisfies

$$\frac{\partial^p}{\partial x_{i_p} \ldots \partial x_{j_p}} \{d[\operatorname{diag}(x_{\alpha_1}, \ldots, x_{\alpha_r}) + K[\alpha_1 \ldots \alpha_r]] - d[\operatorname{diag}(x_{\alpha_1}, \ldots, x_{\alpha_r}) + H[\alpha_1, \ldots, \alpha_r]]\} = 0. \tag{12}$$

If w is a subset of $\alpha_1 < \ldots < \alpha_r$ let $\alpha_1^w < \ldots < \alpha_{r-p}^w$ be the ordered complementary set of w in $\alpha_1 < \ldots < \alpha_r$. In this case the derivative in (12) has the value

$$d[\operatorname{diag}(x_{\alpha_1^w}, \ldots, x_{\alpha_{r-p}^w}) + K[\alpha_1^w, \ldots, \alpha_{r-p}^w]] - d[\operatorname{diag}(x_{\alpha_1^w}, \ldots, x_{\alpha_{r-p}^w}) + H[\alpha_1^w, \ldots, \alpha_{r-p}^w]]. \tag{13}$$

Setting $x_1 = \ldots = x_n = 0$ in (13) we have

$$\phi^0_{i_1 \ldots i_p} = \sum_{1 \leq \alpha_1 < \ldots < \alpha_r \leq n}{}' \{d(K[\alpha_1^w, \ldots, \alpha_{r-p}^w] - d(H[\alpha_1^w, \ldots, \alpha_{r-p}^w])\}, \tag{14}$$

where $\sum'$ indicates that the summation is taken only over those $\alpha_1 < \ldots < \alpha_r$ containing $w=(i_1 < \ldots < i_p)$ as a subset. Returning to (11) we set $x_{i_1} = \ldots = x_{i_p} = t$ and all other $x_j = 0$. Then $\phi(0, \ldots, t, \ldots, t, \ldots 0: U) = t^p \phi^0_{i_1 \ldots i_p} + L_{p-1}(t)$ where L_{p-1} is a polynomial of degree at most $p-1$ in t. Since $\phi(x_1, \ldots, x_n: U) \geq 0$ we conclude, by letting t increase, that $\phi^0_{i_1 \ldots i_p} \geq 0$. Conversely, if $\phi^0_{i_1 \ldots i_p} \geq 0$ and $\phi(0, \ldots, 0, U) \geq 0$, then $\phi(x_1, \ldots, x_n: U) \geq 0$ for all non-negative $x_1, \ldots, x_n$. Let $j_1 < \ldots < j_{n-p}$ denote the set complementary to $i_1 < \ldots < i_p$ in $1, \ldots, n$. Let e_j, $j=1, \ldots, n$, be the unit n-tuple with 1 in position j, 0 elsewhere. We may rewrite (14) as

$$\begin{aligned}
\phi^0_{i_1 \ldots i_p} &= \sum{}' \{dK[\sigma_1, \ldots, \sigma_{r-p}] - dH[\sigma_1, \ldots, \sigma_{r-p}]\} \\
&= \sum{}' \{(C_{r-p}(K)e_{\sigma_1} \wedge \ldots \wedge e_{\sigma_{r-p}}, e_{\sigma_1} \wedge \ldots \wedge e_{\sigma_{r-p}}) \\
&\quad - (C_{r-p}(H)e_{\sigma_1} \wedge \ldots \wedge e_{\sigma_{r-p}}, e_{\sigma_1} \wedge \ldots \wedge e_{\sigma_{r-p}})\} \\
&= \sum{}' (C_{r-p}(F)u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}, u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}) \\
&\quad - \sum{}' (C_{r-p}(G)u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}, \\
&\quad u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}),
\end{aligned} \tag{15}$$

where $\sum'$ indicates that the summation extends over precisely those increasing sequence $\sigma_1 < \ldots < \sigma_{r-p}$ which are subsets of $j_1 < \ldots < j_{n-p}$. Also $u_j = Ue_j$, $j=1, \ldots, n$ is an orthonormal set of vectors (recall that U is unitary). We use an extremal result in [3: theorem 1, p. 525] to conclude from (15) that

$$\phi^0_{i_1 \ldots i_p} \geq E_{r-p}(\lambda_{p+1}(F), \ldots, \lambda_n(F)) - E_{r-p}(\lambda_1(G), \ldots, \lambda_{n-p}(G)) \geq 0.$$

In case $G=I$ we have from (15) again that

$$\phi^0_{i_1 \ldots i_p} = \sum'(C_{r-p}(F) u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}, u_{\sigma_1} \wedge \ldots \wedge u_{\sigma_{r-p}}) - \binom{n-p}{r-p}$$

for all choices of sets of $n-p$ orthonormal vectors $u_1, \ldots, u_{n-p}$, and another application of the above cited extremal result completes the proof.

To proceed to the proof of theorem 2, choose a nonsingular R such that $F=RR^*$, $G=RDR^*$, $D=\text{diag}\ (\lambda_1(F^{-1}G), \ldots, \lambda_n(F^{-1}G))$. Let P be an arbitrary nonsingular matrix and since any $A>0$ is of the form $(PR^{-1})^*(PR^{-1})$ we have that (7) is equivalent to

$$E_r((PR^{-1})^*(PR^{-1})RR^*) \geq E_r((PR^{-1})^*(PR^{-1})RDR^*),$$

$$E_r(P^*P) \geq E_r(P^*PD),$$

or

$$E_r(A) \geq E_r(AD) \text{ for all } A>0. \tag{16}$$

In (16) replace A by VXV^*, V unitary, $X=\text{diag}\ (x_1, \ldots, x_n) \geq 0$ to obtain

$$E_r(X) \geq E_r(XH), \qquad H=V^*DV$$

or

$$tr[C_r(X)(I-C_r(H))]=0, \tag{17}$$

where I is the $\binom{n}{r}$-square identity matrix. It is not difficult to check that (17) holds for all non-negative diagonal X if and only if every diagonal element of $I-C_r(H)$ is non-negative. That is,

$$1-(C_r(V^*DV)e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_r}, \quad e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_r}) \geq 0$$

must hold for all V and all $1 \leq \alpha_1 < \ldots < \alpha_r \leq n$. But this is precisely equivalent to

$$(C_r(D)u_{\alpha_1} \wedge \ldots \wedge u_{\alpha_r}, \quad u_{\alpha_1} \wedge \ldots \wedge u_{\alpha_r}) \leq 1$$

for all orthonormal $u_{\alpha_1}, \ldots, u_{\alpha_r}$. As in the proof of theorem 1 we have finally that (7) holds if and only if (8) does.

To prove theorem 3 it will be convenient to let $\overset{c}{=}$ denote the relation of Hermitian congruence. Then if $A>0$,

$$C_r(I+AF)-C_r(I+AG)=C_r(A)\{C_r(A^{-1}+F)-C_r(A^{-1}+G)\} \overset{c}{=} (C_r(A))^{1/2}[C_r(A^{-1}+F)-C_r(A^{-1}+G)](C_r(A))^{-1/2}.$$

This last matrix has the same roots as the Hermitian matrix $C_r(A^{-1}+F)-C_r(A^{-1}+G)$ and hence (9) is equivalent to

$$C_r(A+F) \geq C_r(A+G) \qquad \text{for all } A>0.$$

Now

$$C_r(A+F)-C_r(A+G)=C_r(A+F)-(C_r(G))^{1/2} C_r(G^{-1/2}AG^{-1/2}+I)(C_r(G))^{1/2} \overset{c}{=} C_r(G^{-1/2}AG^{-1/2}+G^{-1/2}FG^{-1/2})-C_r(G^{-1/2}AG^{-1/2}+I).$$

Thus (9) is equivalent to

$$C_r(A+H)-C_r(A+I) \geq 0 \quad \text{for all } A>0 \tag{18}$$

where $H=G^{-1/2}HG^{-1/2}$.

By a unitary congruence we may assume $H=\text{diag}\ (h_1, \ldots, h_n)$ and by setting $A=\text{diag}\ (x_1, \ldots, x_n) \geq 0$ we see that

$$\prod_{t=1}^{r}(x_{i_t}+h_{i_t}) \geq \prod_{t=1}^{r}(x_{i_t}+1)$$

for any non-negative numbers $x_{i_1}, \ldots, x_{i_r}$. This clearly implies that each $h_t \geq 1$, $t=1, \ldots, n$. Thus, $0 \geq H-I=G^{-1/2}FG^{-1/2}-I \overset{c}{=} F-G$.

Conversely suppose $F-G \geq 0$. Then $H \geq I$ and if we set $B=A+I>0$ we would like to conclude that

$$C_r(A+H)=C_r(B+H-I) \geq C_r(B)=C_r(A+I). \tag{19}$$

But (19) is equivalent to

$$C_r(I+K) \geq C_r(I), \quad K=B^{-1/2}(H-I)B^{-1/2}. \tag{20}$$

After a unitary congruence (20) reduces to

$$C_r(I+\text{diag}\ (k_1, \ldots, k_n)) \geq C_r(I),$$

where $k_\alpha \geq 0$ are the characteristic roots of K. The proof is complete.

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Some Properties of the Empirical Distribution Function of a Random Process

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Let $\{X(t)\}$ be a continuous-time, continuous-in-the-mean, real, strictly stationary random process. If x is a given number, denote by $p_T(x)$ the proportion of the time $X(t)$ exceeds x, $0 \leq t \leq T$. The covariance of $p_T(x_1)$ and $p_T(x_2)$ is obtained. An approximate solution for the variance of a sample quantile is also given. The general results are specialized for Gaussian and χ^2 processes.

1. Introduction

The proportion-of-time distribution plays a very important role in the statistical analysis of time series data. The practice is to plot $p(x)$, the proportion of the time the record, $X(t)$, exceeds x, against x [e.g., 1,2].[1] In general, if $X(t)$ is a wide-sense stationary process and a sample over $0 \leqq t \leqq T$ is available, an estimate, $p_T(x)$, of the probability that $X(t) \geqq x$ is obtained from the sample. In this paper we will derive the covariance of $p_T(x_1)$ and $p_T(x_2)$. An approximate solution for the variance of a sample quantile will also be given. The general results will be specialized for Gaussian and χ^2 processes. The Rayleigh process is a special case of the χ^2 process; however, due to its wide applications, the result for this particular case will also be stated.

2. Some Properties of Time Averages

Let $\{X_1(t), X_2(t)\}$, $t \epsilon R^{(1)}$, be a wide-sense stationary, real, continuous-parameter, continuous-in-the-mean, vector process. t will be called time and $R^{(1)}$ will be taken to be a real line.

For $i=1, 2$, and for all t and s, let

$$\left.\begin{aligned} &EX_i(t) = m_i, \\ &EX_i(t)X_i(t+s) - m_i^2 = R_i(s), \\ &EX_1(t)X_2(t+s) - m_1 m_2 = R_{12}(s), \end{aligned}\right\} \tag{2.1}$$

where EX denotes the mathematical expectation of the random variable X. It will be assumed that all quantities in (2.1) are finite.

When statements are made which apply to either of the processes, or when the properties of a single process are considered, the process will be referred to as $\{X(t)\}$, and the subscripts will be omitted, e.g., $EX(t)=m$, $EX(t)X(t+s)-m^2=R(s)$.

Let T be a positive number, N a positive integer, $d=T/N$, and denote $X_i(kd)$ by $X_i(k)$, $R_i(kd)$ by $R_i(k)$, etc., since there will be little danger of confusion. $\{X_1(k), X_2(k)\}$, $k=0, \pm 1, \pm 2, \ldots$, is, then, a wide-sense stationary, real, discrete-parameter, vector process.

We may consider the discrete-time process and extend the results to the continuous-time process by taking the limit as $N \to \infty$, T fixed; or, we may consider the continuous-time process and obtain results for the discrete-time process by replacing integrals by appropriate summations. In general, we will pursue the latter procedure.

[1] Figures in brackets indicate the literature references at the end of this paper.

Let t_1, t_2 be arbitrary numbers, and T_1, T_2 arbitrary positive numbers. Define

$$\overline{X}_{iT_i}(t_i)=\frac{1}{T_i}\int_{t_i}^{t_i+T_i} X_i(t)\,dt, \qquad i=1,2. \tag{2.2}$$

It will be assumed that the subscripts on X are so chosen that $T_2 \geqq T_1$.

For the discrete-time case, we take k_1, k_2 arbitrary integers; N_1, N_2 arbitrary positive integers, choosing the subscripts on X so that $N_2 \geqq N_1$; and define

$$\overline{X}_{iN_i}(k_i)=\frac{1}{N_i}\sum_{k=k_i+1}^{N_i+k_i} X_i(k), \qquad i=1,2. \tag{2.2'}$$

Now

$$E\overline{X}_{iT_i}(t_i)=m_i, \qquad i=1,2, \tag{2.3}$$

and

$$\operatorname{cov}\{\overline{X}_{1T_1}(t_1), \overline{X}_{2T_2}(t_2)\}=(T_1T_2)^{-1}E\int_{t_1}^{T_1+t_1}\int_{t_2}^{T_2+t_2}[X_1(t')-m_1]\,[X_2(t)-m_2]\,dt\,dt'$$

$$=(T_1T_2)^{-1}\int_{t_1}^{T_1+t_1}\int_{t_2}^{T_2+t_2}R_{12}(t-t')dt\,dt'. \tag{2.4}$$

The interchange of operations of expectation and integration in (2.3) and (2.4) is justified as $EX_i^2(t)<\infty$, $i=1,2$.

We now make the transformation $u=t-t'$, $v=t'$. The Jacobian of the transformation is unity. The limits of u and v are given by

$$t_2-t_1-T_1 \leqq u \leqq t_2-t_1+T_2,$$

$$\max(t_1, t_2-u) \leqq v \leqq \min(T_1+t_1, T_2+t_2-u).$$

Hence (2.4) becomes

$$(T_1T_2)^{-1}\int_{-T_1+s}^{T_2+s}[\min(T_1+t_1, T_2+t_2-u)-\max(t_1, t_2-u)]\,R_{12}(u)\,du,$$

where $s=t_2-t_1$. Now

$$\min(T_1+t_1, T_2+t_2-u)=\begin{cases} T_1+t_1, \text{ when } u \leqq s+T_2-T_1 \\ T_2+t_2-u, \qquad \text{otherwise;} \end{cases}$$

$$\max(t_1,t_2-u)=\begin{cases} t_2-u, \text{ when } u \leqq s, \\ t_1, \text{ otherwise.} \end{cases} \tag{2.5}$$

Replacing t_1 by t, we obtain, for arbitrary t and s,

$$\operatorname{cov}\{\overline{X}_{1T_1}(t), \overline{X}_{2T_2}(t+s)\}=(T_1T_2)^{-1}\Big[\int_{-T_1+s}^{s}(T_1-s+u)R_{12}(u)du+T_1\int_{s}^{T_2-T_1+s}R_{12}(u)du$$

$$+\int_{T_2-T_1+s}^{T_2+s}(T_2+s-u)R_{12}(u)du\Big]. \tag{2.6}$$

Remark 1. Setting $X_1(t)=X_2(t)=X(t)$ in (2.6), we obtain

$$\operatorname{cov}\{\overline{X}_{T_1}(t), \overline{X}_{T_2}(t+s)\}=(T_1T_2)^{-1}\Big[\int_{-T_1+s}^{s}(T_1-s+u)R(u)du$$

$$+T_1\int_{s}^{T_2-T_1+s}R(u)du+\int_{T_2-T_1+s}^{T_2+s}(T_2+s-u)R(u)du\Big]. \tag{2.7}$$

Remark 2. If $T_1 = T_2 = T$,

$$\operatorname{cov}\{\overline{X}_T(t), X_T(t+s)\} = T^{-2}\left[\int_{-T+s}^{s}(T-s+u)R(u)du + \int_{s}^{T+s}(T+s-u)R(u)du\right]. \quad (2.8)$$

Remark 3. Setting $s=0$ in (2.8), we have

$$\operatorname{var}\overline{X}_T(t) = T^{-1}\int_{-T}^{T}\left(1-\frac{|u|}{T}\right)R(u)du. \quad (2.9)$$

From (2.3) to (2.9) we deduce that, for fixed T_1 and T_2, $\{\overline{X}_{1T_1}(t), \overline{X}_{2T_2}(t)\}$ is a wide-sense stationary continuous-time and continuous-in-the-mean process.

For the discrete-time case, we will state only the results corresponding to (2.6) and (2.9). When translating a sum of integrals, such as

$$\int_a^b + \int_b^c$$

into a Riemann sum, we should be careful to note that, whereas b appears twice in the limits of the integrals, the corresponding quantity will appear only once in the sum of two summations. Thus, for arbitrary integers k and s,

$$\operatorname{cov}\{\overline{X}_{1N_1}(k), X_{2N_2}(k+s)\} = (N_1N_2)^{-1}\left[\sum_{u=-N_1+s+1}^{s}(N_1-s+u)R_{12}(u) + \sum_{u=s+1}^{N_2-N_1+s}N_1R_{12}(u) + \sum_{u=N_2-N_1+s+1}^{N_2+s-1}(N_2+s-u)R_{12}(u)\right]. \quad (2.6')$$

Similarly,

$$\operatorname{var}\overline{X}_N(k) = \frac{R(0)}{N} + \frac{2}{N}\sum_{u=1}^{N-1}\left(1-\frac{u}{N}\right)R(u). \quad (2.9')$$

The results (2.9) and (2.9′) are well known [3, p. 80].

3. Empirical Distribution Function

It will be assumed further that $\{X(t)\}$ is a strictly stationary process so that the probability distribution of $X(t_1+h), \ldots, X(t_n+h)$ is the same as that of $X(t_1), \ldots, X(t_n)$ for every selection of $n, t_1, \ldots, t_n$, and h.

Define

$$p(t;x) = \begin{cases} 1, \text{ if } X(t) \geqq x, \\ 0, \text{ otherwise.} \end{cases} \quad (3.1)$$

Then

$$p_T(x) = \frac{1}{T}\int_0^T p(t;x)dt \quad (3.2)$$

is the proportion of the time x is exceeded by $X(t)$ in the time interval $(0, T)$, i.e., $Tp_T(x)$ is the Lebesgue measure of the set $S_x = \{t : X(t) \geqq x,\ 0 \leqq t \leqq T\}$. We note the following:

(1) $p(t; -\infty) = 1$ for all t; hence $p_T(-\infty) = 1$;
(2) $p(t; +\infty) = 0$ for all t; hence $p_T(+\infty) = 0$;
(3) If $x_2 > x_1$, then $X(t) \geqq x_2$ implies $X(t) \geqq x_1$.

S_{x_2} is, therefore, a subset of S_{x_1}, and hence

$$p_T(x_2) \leqq p_T(x_1).$$

Properties (1) to (3) imply that $q_T(x) = 1 - p_T(x)$ is a distribution function. $q_T(x)$ will be called

the *empirical univariate distribution function* of the process $\{X(t)\}$. We will, however, consider $p_T(x)$ rather than $q_T(x)$.

Suppressing x from $p(t; x)$ for the time being, we have

$$\begin{aligned}
&\Pr[p(t_1+h)=1, \ldots, \quad p(t_k+h)=1, \quad p(t_{k+1}+h)=0, \ldots, \quad p(t_n+h)=0] \\
&\quad=\Pr[X(t_1+h)\geqq x, \ldots, \quad X(t_k+h)\geqq x, \quad X(t_{k+1}+h)<x, \ldots, \quad X(t_n+h)<x] \\
&\quad=\Pr[X(t_1)\geqq x, \ldots, \quad X(t_k)\geqq x, \quad X(t_{k+1})<x, \ldots, \quad X(t_n)<x] \\
&\quad=\Pr[p(t_1)=1, \ldots, \quad p(t_k)=1, \quad p(t_{k+1})=0, \ldots, \quad p(t_n)=0],
\end{aligned} \tag{3.3}$$

for every selection of k, n, $t_1, \ldots, t_k, \ldots, t_n$, and h. Hence, for a given x, $\{p(t; x)\}$ is a strictly stationary binomial random process. In fact, it can be shown that, for a given set of values, $x_1, \ldots, x_m$, $\{p(t, x_1), \ldots, p(t; x_m)\}$ is a strictly stationary vector process.

Writing

$$P(x)=\Pr(X(t)\geqq x), \tag{3.4}$$

we have

$$\left.\begin{aligned}
&Ep^r(t; x)=P(x), \quad r=1, 2, \ldots, \\
&\operatorname{var} p(t; x)=P(x)[1-P(x)], \\
&A(s; x)\equiv\operatorname{cov}\{p(t; x), p(t+s; x)\} \\
&\qquad=\Pr\{X(0)\geqq x, X(s)\geqq x\}-P^2(x).
\end{aligned}\right\} \tag{3.5}$$

Let $x_2\geqq x_1$; then

$$\begin{aligned}
A(s; x_1, x_2)&\equiv\operatorname{cov}\{p(t; x_1), p(t+s; x_2)\} \\
&=Ep(t; x_1)p(t+s; x_2)-P(x_1)P(x_2) \\
&=\Pr\{X(t)\geqq x_1, X(t+s)\geqq x_2\}-P(x_1)P(x_2) \\
&=\Pr\{X(0)\geqq x_1, X(s)\geqq x_2\}-P(x_1)P(x_2).
\end{aligned} \tag{3.6}$$

We note the following:

$$Ep_T(x)=P(x)$$

$$A(s; x, x)=A(s; x),$$

$$A(0; x_1, x_2)=P(x_2)[1-P(x_1)].$$

Using (2.6) with $s=0$, $T_1=T_2=T$, we have

$$\left.\begin{aligned}
C(x_1, x_2)&\equiv\operatorname{cov}\{p_T(x_1), p_T(x_2)\}=\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|s|}{T}\right)A(s; x_1, x_2)ds, \\
C(x, x)&=\operatorname{var} p_T(x)=\frac{2}{T}\int_0^T\left(1-\frac{s}{T}\right)A(s; x)ds.
\end{aligned}\right\} \tag{3.7}$$

For the discrete-time case,

$$\left.\begin{aligned}
p_N(x)&=N^{-1}\sum_{k=1}^{N}p(k; x), \\
\operatorname{cov}\{p_N(x_1), p_N(x_2)\}&=N^{-1}P(x_2)[1-P(x_1)]+2N^{-1}\sum_{s=1}^{N-1}(1-s/N)A(s; x_1, x_2), \\
\operatorname{var} p_N(x)&=N^{-1}P(x)[1-P(x)]+2N^{-1}\sum_{s=1}^{N-1}(1-s/N)A(s; x).
\end{aligned}\right\} \tag{3.7'}$$

To estimate $A(s; x_1, x_2)$ and $A(s; x)$, define

$$p(t, t+s; x_1, x_2)=\begin{cases}1 \text{ if } X(t)\geqq x_1,\ X(t+s)\geqq x_2\\ 0 \text{ otherwise;}\end{cases}$$

and use

$$P_T(s; x_1, x_2)=\frac{1}{T-s}\int_0^{T-s} p(t, t+s; x_1, x_2)dt$$

to estimate $\Pr\{X(t)\geqq x_1,\ X(t+s)\geqq x_2\}$.

4. Variance of a Quantile

Given x, $p_T(x)$ is a random variable. In the preceding section we obtained the variance of $p_T(x)$, and the covariance of $p_T(x_1)$ and $p_T(x_2)$. In this section we will obtain an approximation to the variance of a quantile of the empirical distribution function. We suppose that a number q, $0<q<1$, is specified. If U denotes the number such that

$$p_T(U)=q, \tag{4.1}$$

then U is a random variable. Let ξ be the number such that

$$P(\xi)=q, \tag{4.2}$$

i. e., ξ is the qth quantile of the univariate distribution function of the process $[X(t)]$.

In what follows the primes will denote differentiation with respect to the indicated argument. Furthermore, it will be assumed that $P(\xi)$ and $p_T(\xi)$ have derivatives.

Now

$$P(\xi)=p_T(U)\doteq p_T(\xi)+(U-\xi)p'_T(\xi),$$

where $\doteq$ means "approximately equal to;" hence,

$$p_T(\xi)-P(\xi)\doteq -(U-\xi)p'_T(\xi); \tag{4.3}$$

$$0\doteq -E(U-\xi)[p'_T(\xi)-P'(\xi)]-P'(\xi)E(U-\xi).$$

Now

$$Ep'_T(\xi)=E\lim_{h\to 0}\frac{p_T(\xi+h)-p_T(\xi)}{h}=\lim_{h\to 0}\frac{P(\xi+h)-P(\xi)}{h}=P'(\xi);$$

therefore

$$\operatorname{cov}(U, p'_T(\xi)\doteq -P'(\xi)E(U-\xi).$$

As a first approximation we suppose that

$$EU=\xi, \tag{4.4}$$

so that

$$\operatorname{cov}(U, p'_T(\xi))=0. \tag{4.5}$$

As a further approximation suppose that U and $p'_T(\xi)$ are independent. Writing $\delta y=y-Ey$, where y is a random variable, we have

$$\delta p_T(\xi)\doteq -p'_T(\xi)\delta U,$$

$$\operatorname{var} p_T(\xi)\doteq \operatorname{var} U Ep'^2_T(\xi). \tag{4.6}$$

Now

$$Ep_T'^2(\xi)=E\lim_{h,\,k\to 0}\left(\frac{p_T(\xi+h)-p_T(\xi)}{h}\right)\left(\frac{p_T(\xi+k)-p_T(\xi)}{k}\right)$$

$$=E\lim_{h,\,k\to 0}(hk)^{-1}[\delta p_T(\xi+h)-\delta p_T(\xi)+P(\xi+h)-P(\xi)][\delta p_T(\xi+k)-\delta p_T(\xi)+P(\xi+k)-P(\xi)]$$

$$=P'^2(\xi)+\lim_{h,\,k\to 0}(hk)^{-1}[C(\xi+h,\xi+k)-C(\xi+h,\xi)-C(\xi,\xi+k)+C(\xi,\xi)]$$

$$=P'^2(\xi)+\frac{\partial^2 C(\xi_1,\xi_2)}{\partial\xi_1\partial\xi_2}\bigg|_{\xi_1=\xi_2=\xi}.$$

$$\operatorname{var} U\doteq\left[P'^2(\xi)+\frac{\partial^2 C(\xi_1,\xi_2)}{\partial\xi_1\partial\xi_2}\bigg|\xi_1=\xi_2=\xi\right]^{-1}\operatorname{var} p_T(\xi), \tag{4.7}$$

and var $p_T(\xi)$ is given by (3.7). The result (4.7) assumes the existence of various derivatives appearing in the equation. In applying this result it is necessary to look into various approximating assumptions which are used in arriving at (4.7). The approximation (4.5) seems fairly reasonable; but the assumption of independence of U and $p_T(\xi)$ needs careful examination in each individual case.

From (3.7)

$$\frac{\partial^2 C(x_1,x_2)}{\partial x_1\partial x_2}=\frac{1}{T}\frac{\partial^2}{\partial x_1\partial x_2}\int_{-T}^{T}\left(1-\frac{|s|}{T}\right)A(s;x_1,x_2)ds=\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|s|}{T}\right)\frac{\partial^2 A(s;x_1,x_2)}{\partial x_1\partial x_2}\,ds \tag{4.8}$$

in case the interchange of the integration and the partial differentiation is justified. If

$$\int_{-\infty}^{\infty}\left|\frac{\partial^2 A}{\partial x_1\partial x_2}\right|_{x_1=x_2=\xi}ds<\infty,$$

then

$$\frac{\partial^2 C}{\partial x_1\partial x_2}\bigg|_{x_1=x_2=\xi}=O\left(\frac{1}{T}\right);$$

and therefore

$$\operatorname{var} U\doteq\operatorname{var} p_T(\xi)/P'^2(\xi)[1+O(T^{-1})] \tag{4.9}$$

5. Applications

5.1. Gaussian Process

If $\{X(t)\}$ is Gaussian, write

$$R(0)=\sigma^2 \qquad Y(t)=[X(t)-m]/\sigma \qquad \rho(t)=R(t)/\sigma^2.$$

It is known [4, pp. 355–6] that

$$\Pr\{Y(t)\geqq y_1,\quad Y(t+s)\geqq y_2\}=(2\pi)^{-1}[1-\rho^2(s)]^{-1/2}\int_{y_1}^{\infty}\int_{y_2}^{\infty}\exp\left\{-\frac{z_1^2-2\rho(s)z_1z_2+z_2^2}{2[1-\rho^2(s)]}\right\}dz_1dz_2$$

$$=P(y_1)P(y_2)+\sum_{j=1}^{\infty}\rho^j(s)\,\frac{H_{j-1}(y_1)H_{j-1}(y_2)e^{-\frac{y_1^2+y_2^2}{2}}}{2\pi j!}$$

$$=P(y_1)P(y_2)+\sum_{j=1}^{\infty}\rho^j(s)\tau_j(y_1)\tau_j(y_2). \tag{5.1}$$

Here

$$P(y)=(2\pi)^{-1/2}\int_{y}^{\infty}e^{-z^2/2}\,dz;$$

$H_r(x)$ is the Hermite polynomial of degree r given by

$$H_r(x)=e^{x^2/2}(-d/dx)^r e^{-x^2/2}=x^r-\frac{r(r-1)}{2\,1!}x^{r-2}+\frac{r(r-1)(r-2)(r-3)}{2^2\,2!}x^{r-4}-\ldots,$$

and

$$\tau_r(x)=\frac{H_{r-1}(x)\,e^{-x^2/2}}{(2\pi)^{1/2}\,(r!)^{1/2}}$$

is known as the tetrachoric function of order r, and its tables are available [5].

Thus

$$\left.\begin{aligned}A(s;x_1,x_2)&=\sum_{j=1}^{\infty}\rho^j(s)\tau_j\left(\frac{x_1-m}{\sigma}\right)\tau_j\left(\frac{x_2-m}{\sigma}\right),\\ A(s;x)&=\sum_{j=1}^{\infty}\rho^j(s)\tau_j^2\left(\frac{x-m}{\sigma}\right).\end{aligned}\right\}\tag{5.2}$$

Inserting these values in (3.7), we have, since $\rho(-s)=\rho(s)$,

$$\left.\begin{aligned}C(x_1,x_2)&=\frac{2}{T}\int_0^T(1-s/T)\sum_{j=1}^{\infty}\rho^j(s)\tau_j\left(\frac{x_1-m}{\sigma}\right)\tau_j\left(\frac{x_2-m}{\sigma}\right)ds,\\ C(x,x)&=\frac{2}{T}\int_0^T(1-s/T)\sum_{j=1}^{\infty}\rho^j(s)\tau_j^2\left(\frac{x-m}{\sigma}\right)ds.\end{aligned}\right\}\tag{5.3}$$

5.2. Variance of a Quantile

Note that:

$$\frac{d\tau_j(x)}{dx}=(2\pi)^{-1/2}(j!)^{-1/2}\,d/dx\,[e^{-x^2/2}H_{j-1}(x)]=-(j+1)^{1/2}\tau_{j+1}(x).$$

Differentiating the first equation in (5.3) with respect to x_1 and x_2, we get the formal expression

$$\frac{\partial^2 C}{\partial x_1\,\partial x_2}\Big|_{x_1-x_2=\xi}=\frac{2}{\sigma^2 T}\int_0^T(1-s/T)\sum_{j=1}^{\infty}(j+1)\rho^j(s)\,\tau_{j+1}^2\left(\frac{\xi-m}{\sigma}\right)ds.\tag{5.4}$$

If the integral converges, we can evaluate it, and using (4.7) obtain the variance of the sample quantile U.

A special case. If $x=m$,

$$\operatorname{var}p_T(m)=2T^{-1}\int_0^T(1-s/T)\sum_{j=1}^{\infty}\rho^j(s)\,\tau_j^2(0)\,ds.$$

Now

$$\tau_{2j}(0)=0,\qquad j=1,2,\ldots,$$

$$\tau_{2j+1}(0)=\frac{(-1)^j(2j)!}{(2\pi)^{1/2}\,2^j(j!)\,[(2j+1)!]^{1/2}},\qquad j=0,1,\ldots\ .$$

Hence

$$\operatorname{var}p_T(m)=\frac{1}{\pi T}\sum_{j=0}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2\,(2j+1)}\int_0^T(1-s/T)\,\rho^{2j+1}(s)\,ds.\tag{5.5}$$

If U is the sample median, then $\xi=m$, and

$$\frac{\partial^2 C}{\partial x_1\,\partial x_2}\Big|_{x_1=x_2=m}=(\pi\sigma^2 T)^{-1}\int_0^T (1-s/T)\sum_{j=1}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2}\rho^{2j}(s)\,ds$$

$$=(\pi\sigma^2 T)^{-1}\int_0^T (1-s/T)\,[\{1-\rho^2(s)\}^{-1/2}-1]\,ds. \tag{5.6}$$

in case the integral converges. We note that, for large value of j

$$\frac{(2j)!}{2^{2j}(j!)^2}\simeq\frac{1}{(\pi j)^{1/2}}.$$

Hence the series (5.5) will always converge. A sufficient condition for the convergence of the integral in (5.6) is that

$$\lim_{s\to 0} s^b\{1-\rho^2(s)\}^{-1/2}$$

exists for some $b, 0<b<1$. If the limit is nonzero for some $b\geq 1$ then the integral diverges.

Example 1. If $\rho(s)=\rho(-s)=e^{-\lambda s}, \lambda>0, s\geqq 0$,

$$\int_0^T (1-s/T)e^{-j\lambda s}ds=\frac{1}{j\lambda}-\frac{1}{j^2\lambda^2 T}+O(T^{-1}e^{-j\lambda T}),\qquad j=1,2,\ldots.$$

Therefore

$$\operatorname{var} p_T(m)=\frac{1}{\pi\lambda T}\sum_{j=0}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2(2j+1)^2}-\frac{1}{\pi\lambda^2 T^2}\sum_{j=0}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2(2j+1)^3}+O(T^{-1}e^{-\lambda T})$$

$$\doteq 0.35/(\lambda T)-0.33/(\lambda T)^2+O(T^{-1}e^{-\lambda T}).$$

Also

$$\frac{\partial^2 C}{\partial x_1\partial x_2}\Big|_{x_1=x_2=m}=\frac{1}{\lambda\pi\sigma^2 T}\sum_{j=1}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2(2j)}-\frac{1}{\pi\lambda^2\sigma^2 T^2}\sum_{j=1}^{\infty}\frac{(2j)!}{2^{2j}(j!)^2(2j)^2}+O(T^{-1}e^{-\lambda T})$$

$$=\frac{ln2}{\pi\lambda\sigma^2 T}-\frac{0.17}{\pi\lambda^2\sigma^2 T^2}+O(T^{-1}e^{-\lambda T}).$$

Hence, if U is the sample median,

$$\operatorname{var} U\cong\left[\frac{1}{2\pi\sigma^2}+\frac{ln2}{\pi\lambda\sigma^2 T}-\frac{0.17}{\pi\lambda^2\sigma^2 T^2}\right]^{-1}\operatorname{var} p_T(m)$$

$$=2\pi\sigma^2\operatorname{var} p_T(m)\left[1+\frac{2ln2}{\lambda T}-\frac{0.34}{\lambda^2 T^2}\right]^{-1}.$$

Example 2. If $\rho(s)=e^{-\frac{s^2}{2a^2}}$, a similar calculation shows that

$$\operatorname{var} p_T(m)=0.47a/T-0.35a^2/T^2+O\left(T^{-1}e^{-\frac{T^2}{2a^2}}\right).$$

The integral (5.6) does not converge and we fail in evaluating the variance of the sample median by this method. In fact, an examination of the assumptions leading to the equation (4.7) shows that, in this case, $Ep_T'^2(\xi)$ does not exist so that $p_T(\xi)$ is not differentiable.

Example 3. Let $\rho(s)=\cos(\pi s/k)$. Instead of using (5.5) to evaluate $\operatorname{var} p_T(m)$, we proceed somewhat differently. Note that [6, p. 290]

$$\Pr\{Y(t)\geqq 0, Y(t+s)\geqq 0\}=\frac{1}{2}-\frac{1}{2\pi}\cos^{-1}\rho(s),\qquad 0\leqq\cos^{-1}\rho(s)\leqq\pi.$$

Hence

$$\text{var } p_T(m)=\frac{1}{2T}\int_0^T\left(1-\frac{s}{T}\right)ds-\frac{1}{\pi T}\int_0^T\left(1-\frac{s}{T}\right)\cos^{-1}\rho(s)ds.$$

Now

$$\cos^{-1}\rho(s)=\begin{cases}\frac{\pi}{k}(s-2jk), & \text{if } 2jk\leqq s<(2j+1)k, \quad j=0, 1, \ldots,\\ -\frac{\pi}{k}(s-2jk), & \text{if } (2j-1)k\leqq s<2jk, \quad j=1, 2, \ldots.\end{cases}$$

Let $T=lk+T_1$, $l\geqq 0$, $0\leqq T_1<k$. We have

$$\text{var } p_T(m)=\frac{1}{4}+\sum_{j=0}^{l-1}I_j-\frac{1}{kT}\int_{T-T_1}^{T}\left(1-\frac{s}{T}\right)\cos^{-1}\rho(s)ds,$$

where

$$I_{2j}=-\frac{1}{kT}\int_{2jk}^{(2j+1)k}\left(1-\frac{s}{T}\right)(s-2jk)ds=-\frac{k}{2T}+\frac{k^2}{3T^2}+\frac{jk^2}{T^2},$$

$$I_{2j-1}=\frac{1}{kT}\int_{(2j-1)k}^{2jk}\left(1-\frac{s}{T}\right)(s-2jk)ds=-\frac{k}{2T}-\frac{k^2}{3T^2}+\frac{jk^2}{T^2}.$$

Thus, after some simplification, we obtain

$$\text{var } p_T(m)=\begin{cases}\frac{T_1^2}{12T^2}\left(3-\frac{2T_1}{k}\right), & \text{if } l \text{ is even}\\ \frac{1}{12T^2}\left(k^2-3T_1^2+\frac{2T_1^3}{k}\right) & \text{if } l \text{ is odd.}\end{cases}$$

Obviously var $p_T(m)=0$ whenever $T_1=0$, l even, i.e., whenever T is a multiple of $2k$, the period of the process, and also var $p_T(m)\to 0$ as $T\to\infty$.

5.3. χ^2 Process

Let $\{X_i(t)\}$, $i=1, 2, \ldots, n$, be independent stationary Gaussian processes with the same variance, σ^2, and the same autocorrelation function

$$\rho(s)=[EX_i(t)X_i(t+s)-m_i^2]/\sigma^2.$$

Define

$$Y(t)=\sum_{i=1}^{n}\sigma^{-2}\{X_i(t)-m_i\}^2.$$

$\{Y(t)\}$ will be called a χ^2 process as the univariate distribution of $Y(t)$ is a χ^2 distribution with n degrees of freedom which has the density function

$$g(z;\, n/2)=[2^{n/2}\Gamma(n/2)]^{-1}e^{-z/2}z^{n/2-1}, \text{ for } z>0,$$

$$=0 \text{ otherwise.}$$

It is easy to verify that $\{Y(t)\}$ is strictly stationary.

The characteristic function, $\phi(u,v)$, of $[Y(t),Y(t+s)]$ is easily evaluated to be

$$\begin{aligned}\phi(u,v)&=Ee^{iuY(t)+ivY(t+s)}\\&=[1-2iu-2iv-4uv(1-\rho^2(s))]^{-n/2}\\&=(1-2iu)^{-n/2}(1-2iv)^{-n/2}\left[1+\frac{4\rho^2(s)uv}{(1-2iu)(1-2iv)}\right]^{-n/2}\\&=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(n/2+j)}{\Gamma(n/2)\Gamma(j+1)}\frac{\{2\rho(s)\}^{2j}u^jv^j}{(1-2iu)^{n/2+j}(1-2iv)^{n/2+j}};\end{aligned}\tag{5.7}$$

as

$$\left|\frac{4uv}{(1-2iu)(1-2iv)}\right|=\frac{4|uv|}{[(1+4u^2)(1+4v^2)]^{1/2}}<1,$$

the binomial expansion is uniformly convergent.

For a full discussion of the distribution of $[Y(t),Y(t+s)]$ the reader is referred to [7]. A technique of obtaining the probability density function of $[Y(t),Y(t+s)]$ will be presented here, which is also applicable to other distributions which can be expanded in a series of orthogonal polynomials.

Note the following:

(1)
$$g(z;n/2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-tuz}}{(1-2iu)^{n/2}}du,$$

$$\frac{d^j}{dz^j}g(z;n/2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(-iu)^je^{-tuz}}{(1-2iu)^{n/2}}du,$$

(2) It is easy to verify from the definition of Laguerre polynomials [8, Ch. 5] that, for $\alpha>0$, $j\leqq\alpha-1$,

$$\frac{d^j}{dz^j}g(z;a)=g(z;a-j)\frac{\Gamma(j+1)\Gamma(a-j)}{2^j\Gamma(a)}L_j^{(a-j-1)}(z/2),$$

where

$$L_j^{(a)}(z)=\sum_{k=0}^{j}(-1)^k\binom{a+j}{j-k}\frac{z^k}{k!},$$

is the generalized Laguerre polynomial of degree j.

Hence, if $n\geqq 2-n=1$ may be reduced to the Gaussian case discussed in 5.1—, by the inversion of (5.7) the probability density function, $g(z_1,z_2;n/2)$, of $[Y(t),Y(t+s)]$ is given by

$$\begin{aligned}g(z_1,z_2;n/2)&=\sum_{k=0}^{\infty}\frac{[2\rho(s)]^{2j}\Gamma(n/2+j)}{\Gamma(n/2)\Gamma(j+1)}\frac{d^j}{dz_1^j}g(z_1;n/2+j)\frac{d^j}{dz_2^j}g(z_2;n/2+j)\\&=g(z_1;n/2)g(z_2;n/2)\sum_{j=0}^{\infty}\frac{\Gamma(j+1)\Gamma(n/2)}{\Gamma(n/2+j)}\rho^{2j}(s)L_j^{(n/2-1)}(z_1/2)L_j^{(n/2-1)}(z_2/2).\end{aligned}\tag{5.8}$$

Now, using the first equation in (5.8), we obtain

$$\begin{aligned}\Pr\{Y(t)\geqq y_1,Y(t+s)\geqq y_2\}&=\int_{y_1}^{\infty}\int_{y_2}^{\infty}g(z_1;z_2;n/2)dz_2dz_1\\&=P(y_1)P(y_2)+\sum_{j=1}^{\infty}\frac{[2\rho(s)]^{2j}\Gamma(n/2+j)}{\Gamma(n/2)\,\Gamma(j+1)}\frac{d^{j-1}}{dy_1^{j-1}}g(y_1;n/2+j)\frac{d^{j-1}}{dy_2^{j-1}}g(y_2;n/2+j)\\&=P(y_1)P(y_2)+g(y_1;n/2+1)g(y_2;n/2+1)\sum_{j=1}^{\infty}\frac{2n\Gamma(j)\Gamma(n/2+1)\rho^{2j}(s)}{j\Gamma(n/2+j)}\\&\quad L_{j-1}^{(n/2)}(y_1/2)L_{j-1}^{(n/2)}(y_2/2),\end{aligned}\tag{5.9}$$

where

$$P(y)=\int_y^\infty g(z;\ n/2)dz. \tag{5.10}$$

From (3.6) and (5.9), the covariance of $p(t;y_1)$ and $p(t+s;y_2)$,

$$A(s;y_1,y_2)=g(y_1;n/2+1)g(y_2;n/2+1)\sum_{j=1}^{\infty}\frac{2n\Gamma(j)\,\Gamma(n/2+1)\rho^{2j}(s)}{j\Gamma(n/2+j)}\,L_i^{(n/2)}(y_1/2)\,L_i^{(n/2)}(y_2/2).$$

Inserting this value in (3.7), we obtain an expression for the covariance of $p_T(y_1)$ and $p_T(y_2)$. The variance is given by setting $y_1=y_2=y$.

A special case. $n=2$ gives a χ^2 process with two degrees of freedom, i.e., the Rayleigh process. We find

$$\text{cov}\ \{p_T(y_1),p_T(y_2)\}=\frac{8}{T}\,g(y_1;2)g(y_2;2)\sum_{j=1}^{\infty}\frac{1}{j^2}\,L_i^{(1)}{}_{-1}(y_1/2)L_i^{(1)}{}_{-1}(y_2/2)\int_0^T\left(1-\frac{s}{T}\right)\rho^{2j}(s)ds. \tag{5.11}$$

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Another Extension of Heinz's Inequality

Marvin Marcus

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A recent result of Heinz gives bounds on the bilinear form associated with a matrix Q in terms of bounds on the two Hermitian parts of Q. This is extended to certain determinants associated with Q by use of the Grassmann algebra.

Let A and B be non-negative Hermitian (n.n.h.) n-square matrices and let Q be an arbitrary n-square matrix. A recent inequality due to E. Heinz [3][1] states that *if* $A^2-Q^*Q\geq 0$ *and* $B^2-QQ^*\geq 0$, *then*

$$|(Qu, v)|\leq ||A^p u||\ ||B^{1-p}v||, \qquad 0\leq p\leq 1 \tag{1}$$

for any u *and* v. Here $X\geq 0$ means simply that X is non-negative Hermitian. The most recent proof of (1) is found in [1] along with several references to other proofs and extensions.

In this paper the following generalization of (1) will be presented.

THEOREM. *If* $A^2-Q^*Q\geq 0$ *and* $B^2-QQ^*\geq 0$ *and* $u_1, \ldots, u_k, v_1, \ldots, v_k$ *are any* $2k$ *vectors,* $1\leq k\leq n$, *then*

$$|\det\{(Qu_i, v_j)\}|^2\leq \det\{(A^p u_i, A^p u_j)\} \times \det\{(B^{1-p}v_i, B^{1-p}v_j)\} \leq \prod_1^k ||A^p u_i||^2 ||B^{1-p}v_i||^2, \tag{2}$$

where $0\leq p\leq 1$.

In what follows we use certain elementary properties of the kth compound matrix of A, $C_k(A)$, which are found in [5].

LEMMA. *If* $H\geq 0$ *and* $K\geq 0$ *and* $H-K\geq 0$, *then* $C_k(H)-C_k(K)\geq 0$.

PROOF. We may assume K is nonsingular (and hence H will be) by the standard continuity argument. Now $H-K\geq 0$ if and only if $H^{-1/2}(H-K)H^{-1/2} = I-H^{-1/2}KH^{-1/2}\geq 0$. This latter inequality holds if and only if every eigenvalue of $H^{-1/2}KH^{-1/2}$ is at most 1. Now the eigenvalues of $C_k(H^{-1}K)$ are the $\binom{n}{k}$ products taken k at a time of the eigenvalues of $H^{-1}K$. Moreover $H^{-1}K$ has non-negative eigenvalues and hence every eigenvalue of $C_k(H^{-1}K)$ is bounded by 1. Now $C_k(H^{-1}K)=C_k(H^{-1})C_k(K)$ and this latter matrix is similar to $[C_k(H)]^{-1/2}C_k(K)[C_k(H)]^{-1/2}$. Thus

$$C_k(I)-[C_k(H)]^{-1/2}C_k(K)[C_k(H)]^{-1/2}\geq 0$$

and

$$C_k(H)-C_k(K)\geq 0$$

For completeness we next include a very brief and elementary proof of (1) which relies on the fact [4] that $\varphi(\lambda)=\lambda^p$, $0\leq p\leq 1$, is monotone of every order for non-negative λ. A scalar function φ is said to be *monotone of order* n on $a\leq\lambda\leq b$ if whenever $H-K\geq 0$ it follows that $\varphi(H)-\varphi(K)\geq 0$, where H and K are n-square Hermitian with eigenvalues in the interval $a\leq\lambda\leq b$. To see (1) let $Q=UH$ be the polar factorization of Q, $H\geq 0$, U unitary. Then the hypotheses are equivalent to $A^2-H^2\geq 0$, $B^2-(UHU^*)^2\geq 0$. Setting $w=U^*v$ we compute that

$$\begin{aligned}|(Qu, v)|^2&=|(Hu, w)|^2=|(H^pH^{1-p}u,w)|^2\\ &=|(H^pu, H^{1-p}w)|^2\leq (H^{2p}u, u)(H^{2(1-p)}w, w)\\ &=(H^pu, H^pu)(UH^{1-p}U^*v, UH^{1-p}U^*v)\\ &\leq (A^pu, A^pu)(B^{1-p}v, B^{1-p}v)\\ &=||A^pu||^2||B^{1-p}v||^2.\end{aligned}$$

To proceed to the proof of (2) let $u_1\wedge\ldots\wedge u_k$ denote the Grassmann (outer) product [2] of the vectors $u_1, \ldots, u_k$. Then, by the lemma,

$$0\leq C_k(A^2)-C_k(Q^*Q)=C_k^2(A)-C_k^*(Q)C_k(Q)$$

and

$$0\leq C_k^2(B)-C_k(Q)C_k^*(Q).$$

Hence, applying (1) to $C_k(Q)$, $C_k(A)$, and $C_k(B)$ in place of Q, A, and B respectively, we have

$$\begin{aligned}|\det\{(Qu_i, v_j)\}|^2&=|(C_k(Q)u_1\wedge\ldots\wedge u_k, v_1\wedge\ldots\wedge v_k)|^2\\ &\leq \|[C_k(A)]^pu_1\wedge\ldots\wedge u_k\|^2 \|[C_k(B)]^{1-p}v_1\wedge\ldots\wedge v_k\|^2\\ &=\|C_k(A^p)u_1\wedge\ldots\wedge u_k\|^2 \|C_k(B^{1-p})v_1\wedge\ldots\wedge v_k\|^2\\ &=\|A^pu_1\wedge\ldots\wedge A^pu_k\|^2 \|B^{1-p}v_1\wedge\ldots\wedge B^{1-p}v_k\|^2\\ &=\det\{(A^pu_i, A^pu_j)\} \times\det\{(B^{1-p}v_i, B^{1-p}v_j)\}\\ &\leq\prod_{i=1}^k \|A^pu_i\|^2\|B^{1-p}v_i\|^2.\end{aligned}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

This last inequality is an application of the Hadamard determinant inequality to $\{(A^p u_i, A^p u_j)\}$ and $\{(B^{1-p}v_i, B^{1-p}v_j)\}$ and completes the proof.

If $1 \leq i_1 < \ldots < i_k \leq n$ and $1 \leq j_1 < \ldots < j_k \leq n$ are sequences of integers, then $A[i_1, \ldots, i_k | j_1, \ldots, j_k]$ will denote the k-square submatrix of A, $(a_{i_s j_t})$, $s, t = 1, \ldots, k$.

COROLLARY 1. *If $A^2 - Q^*Q \geq 0$ and $B^2 - QQ^* \geq 0$ and $0 \leq p \leq 1$, then*

$$|\det Q[j_1, \ldots, j_k | i_1, \ldots, i_k]|^2 \leq \det A^{2p}[i_1, \ldots, i_k | i_1, \ldots, i_k] \det B^{2(1-p)}[j_1, \ldots, j_k | j_1, \ldots, j_k].$$

PROOF. Let $u_s = e_{i_s}$, $v_s = e_{j_s}$, $s = 1, \ldots, k$ where e_t is the unit vector with 1 in position t, 0 elsewhere.

Let A and B have eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$ respectively. A bound for the left hand member of (2) may be given in terms of these eigenvalues as follows.

COROLLARY 2. *Under the conditions of the theorem*

$$|\det \{(Qu_i, v_j)\}| \leq \prod_1^k \alpha_i^p \beta_i^{1-p} (\det \{(u_i, u_j)\})^{1/2} (\det \{(v_i, v_j)\})^{1/2}. \quad (3)$$

PROOF. Note that

$$\det \{(A^{2p}u_i, u_j)\} = \| C_k(A^p) u_1 \wedge \ldots \wedge u_k \|^2 \leq \prod_{i=1}^k \alpha_i^{2p} \| u_1 \wedge \ldots \wedge u_k \|^2 = \prod_{i=1}^k \alpha_i^{2p} \det \{(u_i, u_j)\}.$$

Applying this to (2) we get (3).

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Mean Motions In Conditionally Periodic Separable Systems[1]

John P. Vinti

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A search of the literature failed to disclose any general statement or proof of a theorem informally current among dynamical astronomers. The present paper gives a proof of the theorem, which states that, in any conditionally periodic separable system the mean frequency n_k of any separation coordinate q_k is equal to $\nu_k \equiv \partial\alpha_1/\partial J_k$. Here α_1 is the energy and J_k is the k'th action variable. The proof is carried out for nonsingular Staeckel systems, so that it is applicable to any nonpolar orbit of an artificial satellite, when the potential leads to separability.

1. Introduction

Conditionally periodic separable systems are commonly illustrated in works on advanced dynamics by the motion of a particle under the joint action of harmonic oscillator forces at right angles. This happens to be a very special case, for which each rectangular coordinate q_k has a constant frequency, equal to the corresponding "fundamental frequency" $\nu_k \equiv \partial\alpha_1/\partial J_k$, where α_1 is the energy and J_k the corresponding action variable.

In a more general system of this type, each generalized coordinate q_k may have a variable frequency, but it appears to be generally believed among dynamical astronomers that the mean frequency of q_k must be equal to ν_k, if the conditionally periodic system is separable. Needing to refer to such a theorem in solving a specific problem, I have searched the literature but have found no explicit statement or proof of it. The present paper is an attempt to furnish such a reference, with a proof sufficiently general to be applicable to all the separable problems that may arise in the gravitational theory of the orbit of a satellite of an oblate planet.

Little of the analysis in this paper can claim to be really new. Much of the pertinent material in the literature, however, is discursive, relatively unavailable, and expressed in notations now unfamiliar to most mathematical physicists. Some of it is inadequate, if not incorrect, especially in the treatment of the periodicity of the q's as functions of the angle variables. Moreover, none of it seems to have been carried out in the shortest and most appropriate way to prove the theorem in question. The present paper attempts to give a concise and correct treatment that will serve this purpose.

It is easy to see why such a theorem should have escaped formal statement and proof. Physicists have not been concerned with mean frequencies of this kind. Dynamical astronomers have been, but ordinarily for nonseparable systems. Until 1957 their only separable problems were the Kepler problem [1][2] for which the coordinate frequencies are all equal to the constant value $\nu_1=\nu_2=\nu_3$, and the problem of two centers, which has remained a curiosity to be found in Charlier's famous book [2], but without application. Since 1957 various potentials have been suggested, by Sterne [3], by Garfinkel [4], and by the author [5], for the gravitational field of an oblate planet, all of which lead to separability and to intermediary satellite orbits with $\nu_1 \neq \nu_2 \neq \nu_3$. Since the solution for these orbits is greatly facilitated by knowledge of the mean coordinate frequencies, it now becomes desirable to have a formal and general proof of the theorem.

It is here convenient to discuss briefly the general plan of the paper, without definitions. The difficulty in proving the theorem arises only when the fundamental frequencies $\nu_1, \ldots, \nu_n$ are incommensurable. If $w_1, \ldots, w_n$ are the angle variables, the plan is first to show that for a conditionally periodic system there exist infinitely many values of the time, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points separated from the initial point $w_1(0), \ldots, w_n(0)$ by integer intervals $\Delta w_k = m_k, \quad k=1, \ldots, n$. This fact follows directly from a theorem of Dirichlet, which is easy to understand and to apply.

To convert this result to q-space, it is necessary to know that q_k is a single-valued, continuous, periodic function of the w's. To show this the author restricts considerations to nonsingular Staeckel systems, proving that for them each q_k is a single-valued differentiable function of $v_k \equiv \int dq_k/p_k$ and each v_k of the angle variables $w_1, \ldots, w_n$. With a careful use of the single-valuedness, the periodic property then follows.

Application of these properties of the q's as functions of the w's shows that, at the values of the time mentioned above, the orbit in q-space then passes arbitrarily closely to points where each q_k would have gone through exactly m_k cycles. The proof of the theorem then follows.

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[2] Figures in brackets indicate the literature references at the end of this paper.

2. Staeckel Systems

If $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ denote the generalized coordinates and momenta of a dynamical system of n degrees of freedom, the system is said to be of the Staeckel type [2, 6, 7, 8, 9] if the Hamiltonian H is given by

$$H=\frac{1}{2}\sum_{k=1}^{n} A_k(q_1, \ldots, q_n)p_k^2+V(q_1, \ldots, q_n), \qquad A_k>0, \quad (k=1, \ldots, n) \tag{1}$$

and if there exist functions $\phi_{ij}(q_i), \psi_i(q_i), \quad i,j=1, \ldots, n$, such that

$$A_k=M_{k1}/\det(\phi_{ij}), \tag{2}$$

$$V=\sum_{k=1}^{n}\psi_k(q_k)A_k, \tag{3}$$

M_{k1} being the cofactor of $\phi_{k1}(q_k)$ in the determinant $\det(\phi_{ij})$.

Conditions (2) and (3) are necessary and sufficient for the separability of a system with such a Hamiltonian.

If we next define the domain Q of the q's as the totality of real values of $q_1, \ldots, q_n$ for which $p_k^2 \geqq 0$, $k=1, \ldots, n$, we may then define a *nonsingular* Staeckel system as one for which $\psi_k(q_k)$ and $\phi_{ki}(q_k)$, $i, k=1, \ldots, n$, exist and are single-valued and for which $\det(\phi_{ij})\neq 0$ anywhere in Q. If we put

$$\Phi\equiv(\phi_{ij}(q_i)) \tag{4}$$

for the Staeckel matrix, then Φ^{-1} exists and is single-valued anywhere in Q; in particular,

$$(\Phi^{-1})_{1k}=A_k \qquad (k=1, \ldots, n) \tag{5}$$

all exist anywhere in Q. (This restriction thus rules out polar orbits from consideration if the right ascension φ is one of the coordinates, since A_3 then becomes infinite on the polar axis.)

The momenta p_k are then given by

$$p_k^2=-2\psi_k(q_k)+2\sum_{i=1}^{n}\phi_{ki}(q_k)\alpha_i, \qquad (k=1, \ldots, n), \tag{6}$$

where the α's are separation constants, α_1 being the energy. (For satellite problems, where $n=3$, $2\alpha_2$ and $2\alpha_3$ are usually denoted by α_2^2 and α_3^2.)

The Hamilton-Jacobi function W is then given by

$$W=\sum_{k=1}^{n}\int_{q_{k0}}^{q_k} p_k dq_k$$

$$=\sum_{k=1}^{n}(\pm)\int_{q_{k0}}^{q_k}\left[-2\psi_k(q_k)+2\sum_{i=1}^{n}\phi_{ki}(q_k)\alpha_i\right]^{1/2} dq_k, \tag{7}$$

where the sign is $\pm$ respectively as $dq_k \gtrless 0$.

3. Conditionally Periodic Staeckel Systems

We call a Staeckel system conditionally periodic if each coordinate is either rotational or librational. A coordinate q_k is rotational if: (i) it is an angle, (ii) with $p_k^2\equiv F_k(q_k)$ there exist positive real numbers c_{1k} and c_{2k} such that $c_{2k}\geqq F_k(q_k)\geqq c_{1k}>0$ for all real values of q_k, and (iii) if

$$F_k(q_k+2\pi)=F_k(q_k). \tag{8}$$

Note that $c_{1k}>0$ rules out asymptotic motions and that the periodicity implied by (8) may reduce to simple constancy. The latter holds, e.g., when q_k is the right ascension ϕ of an artificial satellite, since there does not exist any potential, depending on the right ascension ϕ, which both leads to separability and remains finite on the polar axis.[3] For such a rotational coordinate q_k either $p_k>c_{1k}^{1/2}$ for all q_k or $p_k<-c_{1k}^{1/2}$ for all q_k. In either case

$$v_k(q_k)\equiv\int_{q_{k0}}^{q_k} dq_k/p_k \tag{9}$$

is a single-valued function of q_k, with derivative dv_k/dq_k existing and differing from zero for all values of q_k. Thus q_k is a single-valued differentiable [4] function of v_k.

A coordinate q_i is librational if there exist real numbers a_i, b_i, C_{1i}, and C_{2i} and a real function $G_i(q_i)$ such that

$$p_i^2=(q_i-a_i)(b_i-q_i)G_i(q_i), \tag{10.1}$$

with

$$C_{2i}\geqq G_i(q_i)\geqq C_{1i}>0 \qquad (a_i\leqq q_i\leqq b_i) \tag{10.2}$$

and

$$a_i\leqq q_i(0)\leqq b_i, \tag{10.3}$$

$q_i(0)$ being the initial value of q_i. If we again define v_i by (9), then

$$v_i(q_i)=\pm\int_{q_{i0}}^{q_i}[(q_i-a_i)(b_i-q_i)G_i(q_i)]^{-1/2}dq_i, \tag{11}$$

where the sign is $\pm$ accordingly as $dq_i \gtrless 0$, respectively. Then

$$v_i=\int_{E_{i0}}^{E_i} G_i^{-1/2}dE_i, \tag{12}$$

where the uniformizing variable E_i is defined by the equation

$$2q_i=a_i+b_i+(a_i-b_i)\cos E_i \tag{13}$$

and the requirement that E_i shall always increase as q_i varies. By (12) and (10.2) v_i is then a single-valued function of E_i, with derivative existing and

[3] See the tables in [9], pp. 656, 658, 660.
[4] Hereafter abbreviated to "s.v.d."

nonvanishing for all E_i, so that E_i must be a s.v.d. function of v_i. By (13), however, q_i is a s.v.d. function of E_i, so that, finally, q_i is a s.v.d. function of v_i. Thus in a conditionally periodic Staeckel system any coordinate q_k is a s.v.d. function of the corresponding v_k.

4. The v_k's as Functions of the Angle-Variables w_k

If we now let an increase of 2π be one cycle of a rotational coordinate and a single round trip from a_k to b_k be one cycle of a librational coordinate, we may define the action and angle variables J_k and w_k by

$$J_k \equiv \oint p_k \, dq_k \tag{14}$$

and

$$w_k \equiv \partial W / \partial J_k, \tag{15}$$

where W is now to be considered a function of the q's and the J's, rather than of the q's and the α's. It is well known [10, 11] that J_k and w_k are canonically conjugate, so that

$$\dot{w}_k = \partial \alpha_1 / \partial J_k \equiv \nu_k. \tag{16}$$

If we also define the Jacobi variables B_i by

$$B_i \equiv \partial W / \partial \alpha_i, \tag{17}$$

we obtain

$$B_i = \sum_{k=1}^{n} \frac{\partial W}{\partial J_k} \frac{\partial J_k}{\partial \alpha_i} = \sum_{k=1}^{n} w_k \omega_{ki}, \tag{18}$$

where

$$\omega_{ki} \equiv \partial J_k / \partial \alpha_i \tag{19}$$

Increments $dw_1, \ldots, dw_n$ then lead to

$$dB_i = \sum_{k=1}^{n} (dw_k)\, \omega_{ki} \qquad (i = 1, \ldots, n). \tag{20}$$

But, by (7) and (17),

$$dB_i = \sum_{k=1}^{n} \frac{\partial p_k}{\partial \alpha_i} dq_k = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial p_k^2}{\partial \alpha_i} dq_k / p_k \tag{21}$$

or

$$dB_i = \sum_{k=1}^{n} \phi_{ki}(q_k)\, dv_k, \tag{22}$$

by (21), (6), and (9). Also, by (19), (14), (6), and (9)

$$\omega_{ki} = \oint \phi_{ki}(q_k) dv_k. \tag{23}$$

If we now introduce the matrix Φ, the matrix $\Omega \equiv (\omega_{ki})$, and the row matrices $dv \equiv (dv_1, \ldots, dv_n)$ and $dw \equiv (dw_1, \ldots, dw_n)$, we find from (20) and (22)

$$dv\Phi = dw\Omega. \tag{24}$$

For a nonsingular Staeckel system[5] Φ, Φ^{-1}, and Ω all exist at every point of Q, so that

$$dv = dw\Omega\Phi^{-1} \tag{25}$$

or

$$\partial v_i / \partial w_k = (\Omega\Phi^{-1})_{ki}. \tag{26}$$

Thus each derivative $\partial v_i / \partial w_k$ exists and is single-valued everywhere in Q. Now the J's are all real, by (14). Thus, by (7) and (15), if the p's are all real, then W and the w's are all real; if some of the p's are nonreal, then W is nonreal and so are some of the w's. If all the w's are real, it then follows that all the p's are real, else we should have a contradiction. Thus the domain Q, corresponding to the totality of all real values of the q's for which the p's are all real, also corresponds exactly to the set of all possible real values for all the w's. It therefore follows that each derivative $\partial v_i / \partial w_k$ exists and is single-valued at any point in w-space. Thus each v_k must be a s.v.d. function of $w_1, \ldots, w_n$. In section 3, however, we showed that each q_k is a s.v.d. function of the corresponding v_k. Thus for a conditionally periodic nonsingular Staeckel system each q_k is a s.v.d. function $f_k(w_1, \ldots, w_n)$.

5. Periodic Properties of $q_k = f_k(w_1, \ldots, w_n)$

By (7) and (15)

$$dw_k = \sum_{i=1}^{n} \frac{\partial p_i}{\partial J_k} dq_i. \tag{27}$$

If now each coordinate q_i goes through an integral number m_i of cycles, then by a familiar argument

$$\Delta w_k = \sum_{i=1}^{n} m_i \oint (\partial p_i / \partial J_k) dq_i = \frac{\partial}{\partial J_k} \sum_{i=1}^{n} m_i \oint p_i dq_i = m_k. \tag{28}$$

Thus if each q_k goes through exactly m_k cycles, each w_k increases by the integer m_k. (Note, however, that such simultaneous increases are not always physically possible: this section is thus concerned only with the mathematical properties of the functions $f_k(w_1, \ldots, w_n)$.)

But we are really interested in the inverse problem where each w_k has increased by an integer m_k and we ask what has happened to the q's. Now the q's are uniquely determined by the w's, because of the single-valued property. In the situation of the preceding paragraph where each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$, each angle variable w_k increases by m_k. Since the w's determine the q's uniquely, this has the result that whenever $\Delta w_k = m_k$, $k = 1, \ldots, n$, each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$.

[5] See appendix for examples.

Thus, in the inverse problem, whenever we are given $\Delta w_k = m_k$, $k=1, \ldots, n$, we find that each rotational coordinate q_i must go through exactly m_i cycles and that each librational coordinate q_j must go through some integral number of cycles, τ_j, say. By (28), however, we then find $\tau_j = m_j$. Thus whenever the angle variables w_k are all increased by integer amounts $\Delta w_k = m_k$, each of the functions $q_k = f_k(w_1, \ldots, w_n)$ must go through exactly m_k cycles.

6. Mean Motions

If in a time interval T the number of complete cycles passed through by any coordinate q_k is N_k, the corresponding mean frequency n_k is, by definition

$$n_k = \lim_{T \to \infty} (N_k/T), \tag{29}$$

if the limit exists. We shall now prove that $n_k = \nu_k \equiv \partial\alpha_1/\partial J_k$, $k=1, \ldots, n$, for any conditionally periodic nonsingular Staeckel system.

To do so, note that if $\nu_1, \ldots, \nu_n$ are all commensurable, there exist a positive ν_0 and positive integers $m_1, \ldots, m_n$ such that

$$\nu_k = m_k \nu_0, \qquad (k=1, \ldots, n), \tag{30}$$

where we may choose ν_0 to be the greatest common divisor of the ν_k's. Then, from (16) and (30), during the actual motion,

$$w_k = w_k(0) + m_k \nu_0 t \qquad (k=1, \ldots, n) \tag{31}$$

and in the time interval $T \equiv 1/\nu_0$ we have

$$\Delta w_k = m_k \qquad (k=1, \ldots, n). \tag{32}$$

By section 5 each q_k goes through exactly m_k cycles in this time, so that in this case the motion is truly periodic, with period $1/\nu_0$. The mean frequency of q_k is thus

$$n_k \equiv m_k/T = m_k \nu_0 = \nu_k. \tag{33}$$

If the frequencies $\nu_1, \ldots, \nu_n$ are not all commensurable, we may let

$$\xi_k \equiv \nu_k/\nu_1 \qquad (k=1, \ldots, n) \tag{34}$$

and then at least one of the ξ's will be irrational. Then by (16) and (34), during the actual motion

$$w_k = w_k(0) + \xi_k \nu_1 t. \tag{35}$$

We now use a theorem of Dirichlet [12], which states that if the set of real numbers $\xi_1, \ldots, \xi_n$ has at least one irrational member, then the system of inequalities

$$|\xi_k - m_k/P| < P^{-1-\frac{1}{n}} \qquad (k=1, \ldots, n) \tag{36}$$

has an infinite number of integer solutions for P and the m's. Note that the solutions for P have no upper bound.

To apply this theorem, consider only those values of the time interval T such that $\nu_1 T = P$, where P is an integer that satisfies (36). In this time each w_k increases from its initial value $w_k(0)$ to a final value given by

$$w_k(T) = w_k(0) + P\xi_k, \tag{37}$$

by (35). But by (36)

$$P\xi_k = m_k + \eta_k, \qquad |\eta_k| < P^{-\frac{1}{n}} \tag{38}$$

so that

$$w_k(T) = w_k(0) + m_k + \eta_k. \tag{39}$$

As $\nu_1 T = P$ takes on those larger and larger integer values corresponding to solutions of (36), each η_k approaches zero, by (38). Then, by (39), there exist infinitely many values of T, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points where $\Delta w_k = m_k$, $k=1, \ldots, n$, the m_k's being solutions of (36).

If the initial q's are given by

$$q_k(0) = f_k[w_1(0), \ldots, w_n(0)], \qquad (k=1, \ldots, n), \tag{40}$$

then the values of the q's at any of these times T are given by

$$q_k(T) = f_k[w_1(0) + m_1 + \eta_1, \ldots, w_n(0) + m_n + \eta_n] \qquad (k=1, \ldots, n). \tag{41}$$

As we let $T = P/\nu_1$ assume those larger and larger values already referred to, the q's then approach arbitrarily closely to the values

$$q_k^*(T) = f_k[w_1(0) + m_1, \ldots, w_n(0) + m_n] \qquad (k=1, \ldots, n). \tag{42}$$

This conclusion follows from (38) and the single-valuedness and differentiability of the functions f_k.

Comparison of (40) and (42) then shows that the values $q_k^*(T)$ correspond to $\Delta w_k = m_k$, $k=1, \ldots, n$, and are thus, by section 5, the values that would be reached after each q_k had gone through exactly m_k cycles. Now, by the definition (29), it follows that the mean frequency

$$n_k = \lim_{T \to \infty} (m_k/T), \tag{43}$$

if the limit exists. But $m_k/T = \nu_1 m_k/P$ and, as $T \to \infty$, $\lim(m_k/P) = \xi_k$, by (36). Thus,

$$n_k = \nu_1 \xi_k = \nu_k, \tag{44}$$

by (43) and (34).

Thus, for each coordinate q_k of a conditionally periodic nonsingular Staeckel system, the mean frequency n_k is equal to the corresponding fundamental frequency $\nu_k \equiv \partial\alpha_1/\partial J_k$.

7. Appendix

For theories of satellite orbits, appropriate coordinates are spherical or oblate spheroidal. The corresponding Staeckel matrices and their inverses are, if $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$:

Spherical

$$\Phi=\begin{pmatrix}1 & -r^{-2} & 0\\ 0 & 1 & -\csc^2\theta\\ 0 & 0 & 1\end{pmatrix}\qquad \Phi^{-1}=\begin{pmatrix}1 & r^{-2} & r^{-2}\csc^2\theta\\ 0 & 1 & \csc^2\theta\\ 0 & 0 & 1\end{pmatrix}$$

or, if $x=c[(\xi^2+1)(1-\eta^2)]^{1/2}\cos\phi$, $y=c[(\xi^2+1)(1-\eta^2)]^{1/2}\sin\phi$, $z=c\xi\eta$:

Oblate Spheroidal

$$\Phi=\begin{pmatrix}c^2\xi^2(\xi^2+1)^{-1} & -(\xi^2+1)^{-1} & (\xi^2+1)^{-2}\\ c^2\eta^2(1-\eta^2)^{-1} & (1-\eta^2)^{-1} & -(1-\eta^2)^{-2}\\ 0 & 0 & 1\end{pmatrix}$$

$$\Phi^{-1}=\begin{pmatrix}c^{-2}(\xi^2+1)(\xi^2+\eta^2)^{-1} & c^{-2}(1-\eta^2)(\xi^2+\eta^2)^{-1} & c^{-2}(\xi^2+1)^{-1}(1-\eta^2)^{-1}\\ -\eta^2(\xi^2+1)(\xi^2+\eta^2)^{-1} & \xi^2(1-\eta^2)(\xi^2+\eta^2)^{-1} & (\xi^2+\eta^2)^{-1}[\xi^2(1-\eta^2)^{-1}+\eta^2(1+\xi^2)^{-1}]\\ 0 & 0 & 1\end{pmatrix}$$

For spherical coordinates Φ is most easily written down from the expressions for p_r^2, p_θ^2, and p_ϕ^2 in the Kepler problem [1], with replacement of α_2^2 and α_3^2 by $2\alpha_2$ and $2\alpha_3$. For oblate spheroidal coordinates Φ may be found by comparing eqs (53) and (59.1) of [5] with eq (6) of the present paper.

Note that Φ or Φ^{-1} could fail to exist only when $\sin\theta=0$ or when $\eta^2=1$. This could happen only when the satellite goes over a pole and thus only in a polar orbit. Such a singularity in a polar orbit, however, is to be expected, since nonsingularity of a Staeckel system leads to the q's being differentiable functions of the w's and thus of the time. In a polar orbit, on the other hand, the right ascension $q_3\equiv\phi$ is a discontinuous function of time, being constant except at polar crossings, where it changes by π.

(Paper 65B2-52)

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Some Boundary Value Problems Involving Plasma Media

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A plasma, consisting of a neutral mixture of electrons, ions and molecules, in the presence of a constant magnetic field H_0, possesses a dielectric constant which is in the form of a tensor. Exact solutions of boundary value problems involving such media are obtained for two-dimensional configurations. Explicit results are given for the reflection coefficients of stratified plasma in planar and cylindrical geometry.

1. Introduction

A neutral mixture of positive (and negative) ions and electrons is often described as a plasma. For example, flames, gaseous discharges, strong shock waves, and the ionosphere are various kinds of plasma. The behavior of electromagnetic waves within and in the vicinity of plasma is of great current interest.

Mainly because of the presence of the free electrons the plasma is a dielectric. The collisions of electrons with the ions cause dissipation of energy and thus the dielectric is lossy. A radio wave propagating through the plasma is thus expected to be attenuated. Furthermore, in the presence of a steady magnetic field, the plasma becomes anisotropic so that the dielectric constant is of tensor form and thus, in general, propagation is not reciprocal.

It is the purpose of the present paper to consider the interaction of electromagnetic waves and plasma for a special class of two-dimensional problems. The geometry is chosen so the wave propagation is essentially transverse to the longitudinal axis. Such a restriction permits exact solutions of a number of relevant boundary value problems to be obtained. Since radio waves, particularly microwaves, are being currently employed as a diagnostic tool in probing plasma, it appears that results given below can find application.

An excellent introduction to the theory of propagation of electromagnetic waves in plasma has been given recently by Whitmer [1].[2] A more comprehensive treatment is found in a recent monograph by Ratcliffe [2]. The dielectric behavior and the molecular properties of a plasma have been dealt with by a number of authors [3–6]. In particular, Spitzer [7], has given a very thorough discussion for fully ionized gases.

2. Dielectric Properties of a Plasma

Since the problems to be discussed in the following sections deal with two-dimensional geometry, it is desirable to take the z-axis of the Cartesian or the cylindrical coordinate system in the direction of the applied magnetic field $\vec{H}_0$. In this case the dielectric displacement $\vec{D}$ is related to the electric field $\vec{E}$ by the relation

$$\vec{D}=(\epsilon)\vec{E} \tag{1}$$

where (ϵ) is the tensor dielectric constant. For an implied time factor $\exp(i\omega t)$, it has the form

$$(\epsilon)=\begin{pmatrix}\epsilon'-iq & 0 & 0\\ iq & \epsilon' & 0\\ 0 & 0 & \epsilon''\end{pmatrix} \tag{2}$$

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[1] This work was carried out while the author was on a visit to the Laboratory for Electromagnetic Theory at the Technical University of Denmark.

[2] Figures in brackets indicate the literature references at the end of this paper.

The quantities ϵ', ϵ'', and q are functions of the density of the electrons and the ions and the frequency of collisions between them. They also depend, of course, on the strength of the applied magnetic field H_0 (see appendix).

The case usually considered is where the electromagnetic forces only influence the electrons. Furthermore, the motion of the ions is commonly neglected. For this situation, the properties of the plasma can be approximately described in a macroscopic sense in terms of the following quantities:

ω_0, the (electron) plasma frequency

ω_T, the (electron) gyro frequency

ν, the effective collision frequency (for electrons).

The elements of the dielectric tensor are then given explicitly by

$$\frac{\epsilon'}{\epsilon_0}=1-\frac{i(\nu+i\omega)\,\omega_0^2/\omega}{\omega_T^2+(\nu+i\omega)^2} \tag{3}$$

$$\frac{q}{\epsilon_0}=\frac{-\omega_T\omega_0^2/\omega}{\omega_T^2+(\nu+i\omega)^2} \tag{4}$$

$$\frac{\epsilon''}{\epsilon_0}=1-\frac{i\omega_0^2}{(\nu+i\omega)\omega} \tag{5}$$

The preceding can be generalized to include the influence of heavy ions by simply adding a summation prefix to the ratios on the right-hand side of equations (3), (4), and (5). Then, in each term, the appropriate value of ω_0, ν, and ω_T must be employed. This approach is valid for a weakly ionized medium and has been employed by Hines [8]. In the case of a plasma consisting of a neutral mixture of electrons, one type of heavy ions and a relatively large number of neutral molecules, the elements of the dielectric tensor have the form

$$\frac{\epsilon'}{\epsilon_0}=1-\frac{i(\nu+i\omega)\omega_0^2/\omega}{\omega_T^2+(\nu_e+i\omega)^2}-\frac{i\,(\nu_i+i\omega)\,\hat{\mu}\omega_0^2/\omega}{\hat{\mu}^2\omega_T^2+(\nu_i+i\omega)^2}, \tag{6}$$

$$\frac{q}{\epsilon_0}=-\frac{\omega_T\omega_0^2/\omega}{\omega_T^2+(\nu_e+i\omega)^2}+\frac{^2\omega_T\omega_0^2/\omega}{\hat{\mu}^2\omega_T^2+(\nu_i+i\omega)^2}, \tag{7}$$

$$\frac{\epsilon''}{\epsilon_0}=1-\frac{i\omega_0^2}{(\nu_e+i\omega)\omega}-\frac{i\hat{\mu}\omega_0^2}{(\nu_i+i\omega)\omega}. \tag{8}$$

where

$$\hat{\mu}=\frac{m_e}{m_i}=\frac{\text{mass of electron}}{\text{mass of ion}}$$

and ν_e and ν_i are the collision frequencies of the electrons and the ions, respectively, with the uncharged molecules. The effect of collisions between the electrons and the ions is neglected. ω_0 and ω_T are the (electron) plasma and (electron) gyro frequencies. Since the charge of the ions is equal and opposite to the electrons, it follows that $\hat{\mu}\omega_0$ and $-\hat{\mu}\omega_T$ are the (ion) plasma and (ion) gyro frequencies, respectively.

It can be seen that if $\nu_i<<\omega$, the denominators in the second factors for ϵ' and q behave as $\hat{\mu}^2\omega_T^2-\omega^2$. Thus, at frequencies near the (ion) gyro frequency, the influence of the ions may be significant even though the mass ratio $\hat{\mu}$ is very small. However, when collisions are not negligible, the effect of the ions is usually quite small. In particular, if $\nu_i>>\omega$ and $\nu_e>>\omega$, this fact can be demonstrated by rewriting the elements of the dielectric tensor in the form

$$\frac{\epsilon'}{\epsilon_0}=1-\frac{i\nu_e\omega_0^2/\omega}{\omega_T^2+\nu_e^2}\left[1+\frac{\nu_i}{\nu_e}\,\hat{\mu}\,\frac{\omega_T^2+\nu_e^2}{\hat{\mu}^2\omega_T^2+\nu_i^2}\right] \tag{9}$$

$$\frac{q}{\epsilon_0}=-\frac{\omega_T\omega_0^2/\omega}{\omega_T^2+\nu_e^2}\left[1-\hat{\mu}^2\,\frac{\omega_T^2+\nu_e^2}{\hat{\mu}^2\omega_T^2+\nu_i^2}\right] \tag{10}$$

$$\frac{\epsilon''}{\epsilon_0}=1-i\,\frac{\omega_0^2}{\nu_e\omega}\left[1+\frac{\nu_e}{\nu_i}\,\hat{\mu}\right] \tag{11}$$

since the ratio $(\omega_T^2+\nu_e^2)/(\hat{\mu}^2\omega_T^2+\nu_i^2)$ is of the order of unity in most cases of practical interest. The square bracket terms may be replaced by unity since $\hat{\mu}$ is of the order of 10^{-4}.

When the plasma becomes strongly ionized, the situation is more complicated. The formalism is given in the appendix where the relative number of ions and electrons is unrestricted However, in the case of a fully ionized plasma the elements of the dielectric tensor are given by [9, 10]

$$\frac{\epsilon'}{\epsilon_0}=1+\frac{\omega_0^2(1+\hat{\mu})[\omega^2-\hat{\mu}\omega_T-i\omega\nu(1+\hat{\mu})]}{\Delta} \tag{12}$$

$$\frac{q}{\epsilon_0}=\frac{\omega\,\omega_T\omega_0^2(\hat{\mu}^2-1)}{\Delta} \tag{13}$$

$$\frac{\epsilon''}{\epsilon_0}=1-\frac{\omega_0^2(1+\hat{\mu})}{\omega\,[\omega-i\nu(1+\hat{\mu})]} \tag{14}$$

where

$$\Delta=\omega^2\omega_T^2(1-\hat{\mu})^2-[\omega^2-\hat{\mu}\omega_T^2-i\omega\nu(1+\hat{\mu})]^2.$$

As before, $\hat{\mu}$ is the ratio of the masses of the electron and the ion. Also, ω_0 and ω_T are the (electron) plasma and (electron) gyro frequencies, respectively. In this case, however, ν is the collision frequency between the electrons and the ions which have equal and opposite charge.

It can be immediately seen that if $\hat{\mu}$ is set equal to zero, these expressions reduce to the form of equations (3), (4), and (5), which are derived under somewhat different conditions. It can also be seen that equations (12), (13), and (14) have the same form as (6), (7), and (8) if $\nu=\nu_e=\nu_i=0$. However, when collisions are non-negligible there is a fundamental difference between a weakly and a fully ionized plasma. This is a consequence of the coupling between the equations of motion for the electrons and the ions (see appendix).

Under the very reasonable approximation that $1\pm\hat{\mu}$ can be replaced by 1, it readily follows that eqs (12), (13), and (14) can be written in the form

$$\frac{\epsilon'}{\epsilon_0}=1-\frac{i(\bar{\nu}+i\omega)\omega_0^2/\omega}{\omega_T^2+(\bar{\nu}+i\omega)^2} \tag{15}$$

$$\frac{q}{\epsilon_0}=-\frac{\omega_T\omega_0^2/\omega}{\omega_T^2+(\bar{\nu}+i\omega)^2} \tag{16}$$

$$\frac{\epsilon''}{\epsilon_0}=1-\frac{i\omega_0^2}{(\nu+i\omega)\omega} \tag{17}$$

where $\bar{\nu}=\nu-i\hat{\mu}\omega_T^2/\omega=\nu\left(1-i\dfrac{\hat{\mu}\omega_T^2}{\nu\omega}\right)$.

The quantity $\bar{\nu}$ could be described as an effective (complex) collision frequency. It should be noted that only the elements ϵ' and q of the dielectric tensor involve $\bar{\nu}$; the remaining element ϵ'' is not changed.

It is now evident that the motion of the ions, for a fully ionized gas, can be neglected only if

$$\hat{\mu}\omega_T^2<<\nu\omega.$$

For low frequencies and/or low collision frequencies this condition may well be violated.

3. Field Equations

Maxwell's equations in a source free region with a (tensor) dielectric constant (ϵ) and a (scalar) magnetic permeability μ are given by

$$i(\epsilon)\omega\vec{E}=\text{curl}\,\vec{H} \tag{18}$$

and

$$-i\mu\omega\vec{H}=\text{curl}\,\vec{E}. \tag{19}$$

It is desirable to write the first of these in the form

$$i\omega\vec{E}=(\epsilon^{-1})\ \text{curl}\,\vec{H},$$

where (ϵ^{-1}) is the inverse of the dielectric tensor. It is not difficult to show that

$$\epsilon_0(\epsilon^{-1})=\begin{pmatrix} M & -iK & 0 \\ iK & M & 0 \\ 0 & 0 & \epsilon_0/\epsilon'' \end{pmatrix}, \tag{20}$$

where

$$M=\frac{\epsilon'\epsilon_0}{(\epsilon')^2-q^2} \qquad \text{and} \qquad K=-\frac{q\epsilon_0}{(\epsilon')^2-q^2}.$$

This formula is quite general. The only restriction is that the z-axis is to be taken in the direction of the impressed magnetic field. In the following it is assumed that the fields do not vary in the z-direction. In terms of Cartesian coordinates, Maxwell's equations are then written

$$i\epsilon_0\omega E_x=M\frac{\partial H_z}{\partial y}+iK\frac{\partial H_z}{\partial x} \tag{21}$$

$$i\epsilon_0\omega E_y=iK\frac{\partial H_z}{\partial y}-M\frac{\partial H_z}{\partial x} \tag{22}$$

$$i\epsilon_0\omega E_z=(\epsilon_0/\epsilon'')\left[\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right] \tag{23}$$

$$-i\mu_0\omega H_x=N\frac{\partial E_z}{\partial y} \tag{24}$$

$$-i\mu_0\omega H_y=-N\frac{\partial E_z}{\partial x} \tag{25}$$

$$-i\mu_0\omega H_z=N\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right] \tag{26}$$

where $N=\mu_0/\mu$.

It is a relatively simple matter to eliminate the transverse component of the fields from the preceding six equations. Thus

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{k^2}{MN}\right]H_z=0 \tag{27}$$

and

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{k^2\epsilon''}{N\epsilon_0}\right]E_z=0, \tag{28}$$

where $k=(\epsilon_0\mu_0)^{1/2}\,\omega=2\pi/\text{wavelength}$. These latter two equations are valid only if the magnetic permeability and the elements of the dielectric tensor are constant for at least a given region.

The fact that H_z and E_z individually satisfy a wave equation means that any solution to our problem can be regarded as the linear combination of two partial solutions. In the first of these, $E_z=0$ and in the second, $H_z=0$. Thus without any subsequent loss of generality, attention can be restricted to these cases. It should be emphasized that this decomposition into independent partial fields is valid only when the derivatives with respect to z are zero. As we shall see, the solution for $H_z=0$ is relatively trivial since the constant magnetic field H_0 then has no influence (at least within the limits of magneto-ionic theory).

4. Reflection Coefficient for a Plane Boundary Between Free Space and Plasma

With respect to the Cartesian coordinate system, a homogeneous plasma occupies the space $y>0$ and while $y<0$ corresponds to free space, the constant and uniform magnetic field is parallel to the z-axis. A plane wave with harmonic time dependence (i.e., exp $(i\omega t)$) is incident from below as indicated in figure 1.

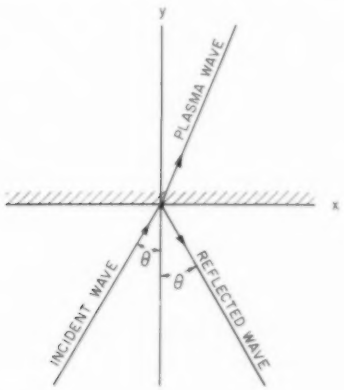

FIGURE 1. *The coordinate system for reflection at a plane interface between a homogeneous plasma ($y>0$) and free space ($y<0$).*

The constant magnetic field $\vec{H_0}$ is along the direction of the positive z-axis (out of the paper).

The angle of incidence is θ (measured to the negative y-axis) and the wave is polarized such that its magnetic field has only a component in the z-direction, denoted H_z^{inc}. Thus,

$$H_z^{inc}=h_0\exp(-ikCy)\exp(-ikSx), \tag{29}$$

where $C=\cos\theta$, $S=\sin\theta$ and h_0 is a constant.

Since the reflected field H_z^{ref} is a solution of the free-space wave equation and is to have the same dependence with x as the incident wave, it must be of the form

$$H_z^{ref}=h_0\,R\exp(ik\,C\,y)\exp(-ik\,S\,x), \tag{30}$$

where R is by definition the reflection coefficient.

With similar reasoning, the solution for the plasma (i.e., $y>0$) must have the form

$$H_z=f(y)\exp(-ik\,S\,x), \tag{31}$$

where $f(y)$ is some function of y. Furthermore, since H_z is to satisfy eq (27), it follows that $f(y)$ must satisfy

$$\left[\frac{\partial^2}{\partial y^2}+k^2\left(\frac{1}{MN}-S^2\right)\right]f(y)=0. \tag{32}$$

Solutions are of the form

$$\exp\left[\pm ik\left[(MN)^{-1}-S^2\right)]^{1/2}y\right]. \tag{33}$$

Since the plasma extends to $y=+\infty$, the solution corresponding to the negative sign in the exponent is pertinent. Therefore, the transmitted wave has the form

$$H_z=h_0\ T\exp\left[-ik\left[(MN)^{-1}-S^2\right]^{1/2}y\right]\exp\left(-ik\,S\,x\right), \tag{34}$$

where T is by definition a transmission coefficient.

The boundary conditions are that the tangential components of the fields in the free space and in the plasma are to be continuous at $y=0$. Continuity of H_z leads to

$$1+R=T, \tag{35}$$

and continuity of E_x, by virtue of eq (21), leads to

$$C\,(1-R)=T\left[\left[(MN)^{-1}-S^2\right]^{1/2}M+iKS\right]. \tag{36}$$

Thus,

$$R=\frac{C-\Delta}{C+\Delta}\qquad\text{and}\quad T=\frac{2C}{C+\Delta}, \tag{37}$$

where

$$\Delta=M\left[(MN)^{-1}-S^2\right]^{1/2}+iKS. \tag{38}$$

For an electron plasma where the motions of the ions are neglected, it is possible to write Δ in the following form:

$$\Delta=\frac{\left[C^2+\dfrac{1+iL}{iL-L^2-\gamma^2}\right]^{1/2}(iL-L^2-\gamma^2)-i\gamma S}{(1+iL)^2-\gamma^2}, \tag{39}$$

where $L=(\nu+i\omega)\,\omega/\omega_0^2$, and $\nu=\omega_T\,\omega/\omega_0^2$. We have also set $N=1$ (i.e., $\mu=\mu_0$) although a plasma may be slightly diamagnetic.

The reflection coefficient, essentially in this form, was derived by Barber and Crombie [11] where the homogeneous electron plasma was to be an idealized representation for the ionosphere. Because of the assumption of a purely transverse magnetic field H_0, the horizontal direction of propagation is along the magnetic equator. For propagation from east-to-west, S is positive, while for propagation from west-to-east S is negative. γ is then a positive real quantity.

For applications at low and very low radio frequencies, $\nu>>\omega$ so that to a good approximation

$$L\cong\omega/\omega_r\qquad\text{where}\qquad\omega_r=\omega_0^2/\nu.$$

Some numerical results based on eq (37) are available [12].

5. Reflection from a Stratified Plasma

We shall now undertake to generalize the previous result to a plasma medium which is stratified in layers all parallel to the free-space interface at $y=0$. The situation is shown in figure 2, where P parallel layers are indicated. The pth layer from the bottom is of thickness l_p and its electrical properties are described by M_p, N_p, and K_p. The index p ranges from 1 to P. It should also be noted that $l_P=\infty$.

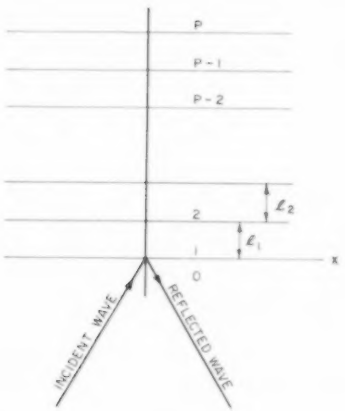

FIGURE 2. *Reflection from a stratified plasma.*

Again taking the incident wave to be polarized with its magnetic vector parallel to the z-axis, it is seen that the field for $y<0$ has the form

$$H_z=h_0\,[\exp\,(-ik\,Cy)+R\,\exp\,(ik\,Cy)]\,\exp\,(-ik\,Sx). \tag{40}$$

The problem is to find an expression for R which involves the properties of the individual layers. It is possible, of course, to formally extend the results for the semi-infinite case by writing the solution in each layer as a linear combination of the two elementary forms given by (33). The two unknown coefficients for each layer are then found from the two boundary conditions at each plane interface. The resulting $2N$ linear equations may then be solved in a straightforward but a very tedious manner for any specified but finite value of N. The resultant solution can be found in a more systematic way if the analogy with Schelkunoff's [13] theory of nonuniform transmission lines is exploited. We use this method here.

The wave impedances for the pth layer are defined by

$$K_p^+=-\frac{E_x^+}{H_z^+} \qquad \text{and} \qquad K_p^-=\frac{E_x^-}{H_z^-}. \tag{41a}$$
$$\tag{41b}$$

The superscript $+$ signifies that the fields vary with y according to the factor $\exp\,[-i\beta_p y]$ where

$$\beta_p=k\,[(M_p\,N_p)^{-1}-S^2]^{1/2},$$

whereas the superscript $-$ signifies that the fields vary with y according to the factor $\exp\,[i\beta_p\,y]$. In the present case, the superscript $+$ signifies a wave traveling in the positive y-direction (i.e., away from the interface) and the $-$ signifies a wave traveling in the negative y-direction.

From eqs (41a) and (41b) it readily follows that the wave impedances are

$$K_p^+=\eta_0\,(M_p\,\beta_p+i\,K_p\,S) \tag{42}$$

and

$$K_p^-=\eta_0\,(M_p\,\beta_p-i\,K_p\,S) \tag{43}$$

where $\eta_0=(\mu_0/\epsilon_0)^{1/2}\cong 120\,\pi$. The index p ranges from 1 to P. Because of the quantity K_p it is seen that K_p^+ and K_p^- are not equal as they would be in an isotropic medium.[3]

The reflection coefficients at the interface between the $(P-1)$th and the Pth layer are now defined by

$$R_{P-1}=\frac{H_z^-}{H_z^+} \qquad \text{and} \qquad r_{P-1}=\frac{E_x^-}{E_x^+}, \tag{44}$$

[3] No confusion should arise between the symbols K_p^+ and K_p^- and K_p, since the superscripted quantities are used only for the wave impedances.

where the field components are evaluated within the $(P-1)$th layer. Thus,

$$R_{P-1}=\frac{K_{P-1}^{+}-K_{P}^{+}}{K_{P-1}^{-}+K_{P}^{+}} \quad \text{and} \quad r_{P-1}=\frac{1/K_{P-1}^{+}-1/K_{P}^{+}}{1/K_{P-1}^{-}+1/K_{P}^{+}}. \tag{45}$$

The reflection coefficients at the interface between the $(P-2)$th and $(P-1)$th layer are then

$$R_{P-2}=\frac{K_{P-2}^{+}-Z_{P-1}}{K_{P-2}^{-}+Z_{P-1}} \quad \text{and} \quad r_{P-2}=\frac{1/K_{P-2}^{+}-1/Z_{P-1}}{1/K_{P-2}^{-}+1/Z_{P-1}}, \tag{46}$$

where Z_{P-1} is the impedance seen at the $(P-1)$th interface. From analogy with transmission line theory

$$Z_{P-1}=K_{P-1}^{+}\frac{1+r_{P-1}\exp(-i2\beta_{P-1}l_{P-1})}{1+R_{P-1}\exp(-i2\beta_{P-1}l_{P-1})}, \tag{47}$$

where r_{P-1} and R_{P-1} are given explicitly by eq (44). Now, in general,

$$Z_{p}=K_{p}^{+}\frac{1+r_{p}\exp(-i2\beta_{p}l_{p})}{1+R_{p}\exp(-i2\beta_{p}l_{p})}, \tag{48}$$

so that the process may be continued until $p=1$, whence

$$R_{1}=\frac{K_{1}^{+}-Z_{2}}{K_{1}^{-}+Z_{2}}. \tag{49}$$

Finally, for the bottom interface

$$R_{0}=\frac{K_{0}^{+}-Z_{1}}{K_{0}^{-}+Z_{1}}, \tag{50}$$

which may be rewritten

$$R=\frac{C-\Delta}{C+\Delta} \tag{51}$$

where $\Delta=Z_1/\eta_0$ since $K_0^{+}=K_0^{-}=\eta_0 C$.

For the special case of a two-layered plasma (i.e., $l_2=\infty$), the explicit expression for Δ becomes

$$\Delta=(M_1\beta_1+iK_1S)\frac{1+r_1\exp(-i2\beta_1 l_1)}{1+R_1\exp(-i2\beta_1 l_1)}, \tag{52}$$

with

$$R_1=\frac{(M_1\beta_1+iK_1S)-(M_2\beta_2+iK_2S)}{(M_1\beta_1-iK_1S)+(M_2\beta_2+iK_2S)} \tag{53}$$

and

$$r_1=\frac{1/(M_1\beta_1+iK_1S)-1/(M_2\beta_2+iK_2S)}{1/(M_1\beta_1-iK_1S)+1/(M_2\beta_2+iK_2S)}. \tag{54}$$

The limiting case of a homogeneous plasma is recovered by letting $l_1\to\infty$, whence

$$\Delta=M_1\beta_1+iK_1S,$$

which is identical to eq (38) after dropping the subscript 1.

6. Scattering from a Cylindrical Plasma Column

In certain applications the plasma may be in the form of a cylindrical column. Examples are the ionization associated with meteor trails in the upper atmosphere and the ionization associated with the shock wave emanating from an exploding wire. In this section expressions for the reflected or scattered fields are derived under the assumption that the ionized column is infinite in length. First, the column is assumed to be homogeneous, but later the solution is generalized to allow for variation of the plasma properties in the radial direction.

Choosing a conventional cylindrical coordinate system (ρ, ϕ, z), a homogeneous plasma column occupies the space $\rho<a$. A constant magnetic field, H_0, exists through the plasma

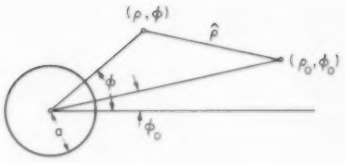

FIGURE 3. *A magnetic line source at* (ρ_0, ϕ_0) *in the presence of a homogeneous cylindrical plasma column.*

The constant magnetic field H_0 is directed along the positive z-axis (out of the paper).

column and it is directed along the z-axis. The medium surrounding the cylinder, $\rho > a$, is taken to be free space. The source of the field is taken to be a line source located at $\rho = \rho_0$ and $\phi = \phi_0$ and running parallel to the z-axis and the plasma column. The line source radiates a cylindrical wave polarized so that its magnetic field has only a z-component.

The adoption of a line source rather than a plane wave source has the advantage of obtaining a more general solution. When $\rho_0 \to \infty$, the incident wave has a plane front but in most practical applications the incident wave front is curved corresponding to a finite value of ρ_0.

The primary or incident field is given by [14]

$$H_z^{\text{inc}} = \frac{\epsilon_0 \omega I}{4} H_0^{(2)}(k\hat{\rho}) \tag{55}$$

where I is the strength of the line source (actually, it is the magnetic current), and where $H_0^{(2)}$ is the Hankel function of order zero of the second kind, and

$$\hat{\rho} = [\rho_0^2 + \rho^2 - 2\rho\,\rho_0 \cos(\phi - \phi_0)]^{1/2}.$$

Employing an additional theorem for $H_0^{(2)}(k\hat{\rho})$, this can be written

$$H_z^{\text{inc}} = \frac{\epsilon_0 \omega I}{4} \sum_{m=-\infty}^{+\infty} H_m^{(2)}(k\rho_0)\, J_m(k\rho)\, e^{-im(\phi - \phi_0)} \tag{56}$$

for $\rho < \rho_0$, where $H_m^{(2)}$ and J_m are Hankel and Bessel functions of order m, respectively. When $\rho > \rho_0$, $k\rho_0$ is to be interchanged with $k\rho$. The ϕ-component of the primary magnetic field is then given by

$$E_\phi^{\text{inc}} = \frac{iI}{4} \sum_{m=-\infty}^{+\infty} H_m^{(2)}(k\rho_0)\, kJ_m'(k\rho)\, e^{-im(\phi - \phi_0)}, \tag{57}$$

where the prime indicates a differentiation with respect to the argument $k\rho$.

Since the scattered or reflected field H_z^s outside the column is a solution of the wave equation, it can be written in the form [14]

$$H_z^s = \frac{\epsilon_0 \omega I}{4} \sum_{m=-\infty}^{+\infty} B_m H_m^{(2)}(k\rho_0)\, H_m^{(2)}(k\rho)\, e^{-im(\phi - \phi_0)} \tag{58}$$

where B_m is an undetermined coefficient. The ϕ component of the scattered electric field is then

$$E_\phi^s = \frac{iI}{4} \sum_{m=-\infty}^{+\infty} B_m H_m^{(2)}(k\rho_0) kH_m^{(2)\prime}(k\rho) e^{-im(\phi - \phi_0)}. \tag{59}$$

For the region $\rho < a$, the magnetic field H_z satisfies

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{k^2}{MN}\right] H_z = 0, \tag{60}$$

which is just the wave equation. Therefore, we may write

$$H_z = \frac{\epsilon_0 \omega I}{4} \sum_{m=-\infty}^{+\infty} A_m H_m^{(2)}(k\rho_0) J_m(\beta\rho) e^{-im(\phi - \phi_0)} \tag{61}$$

where $\beta=k(MN)^{-1}$, and A_m is an undetermined coefficient. Since

$$i\epsilon_\phi\omega E_\phi=iK\frac{\partial H_z}{\rho\partial\phi}-M\frac{\partial H_z}{\partial\rho}, \tag{62}$$

it follows that

$$E_\phi=\frac{iI}{4}\sum_{m=-\infty}^{+\infty}A_mH_m^{(2)}(k\rho_0)\left[-\frac{mK}{\rho}J_m(\beta\rho)+M\beta J_m'(\beta\rho)\right]e^{-im(\phi-\phi_0)}. \tag{63}$$

The boundary conditions at the surface of the plasma cylinder may be written

$$\left.\begin{aligned}H_z&=H_z^{\text{inc}}+H_z^s\\E_\phi&=E_\phi^{\text{inc}}+E_\phi^s\end{aligned}\right\}\rho=a. \tag{64}$$

Application of these leads readily to

$$A_m=\frac{J_m(ka)+B_mH_m^{(2)}(ka)}{J_m(\beta a)} \tag{65}$$

and

$$B_m=-\left[\frac{\left(\frac{M}{N}\right)^{1/2}\frac{J_m'(\beta a)}{J_m(\beta a)}-\frac{mK}{ka}-\frac{J_m'(ka)}{J_m(ka)}}{\left(\frac{M}{N}\right)^{1/2}\frac{J_m'(\beta a)}{J_m(\beta a)}-\frac{mK}{ka}-\frac{H_m^{(2)\prime}(ka)}{H_m^{(2)}(ka)}}\right]\frac{J_m(ka)}{H_m^{(2)}(ka)}. \tag{66}$$

This is the exact solution of the problem. If the constant magnetic field H_0 is removed the results are identical to that of a dielectric cylinder in the presence of a line source [15]. In this limiting case, $K=0$, $M=\epsilon_0/\epsilon$ and $N=\mu_0/\mu$.

7. Scattering from a Cylindrically Stratified Plasma

The preceding results are now generalized to a cylindrical column of ionization which consists of P concentric layers. The situation is illustrated in figure 4. The incident cylindrical wave again emanates from a line source at (ρ_0, ϕ_0) and the field is to be observed at $(\rho, \phi.)$ The wave impedances may now be defined by

$$K_{m,p}^{+}=-\frac{E_{\phi,m}^{+}}{H_{z,m}^{+}}\qquad\text{and}\qquad K_{m,p}^{-}=\frac{E_{\phi,m}^{-}}{H_{z,m}^{-}}, \tag{67}$$

where $+$ and $-$ refer to the two independent wave solutions, proportional to $J_m(\beta_p\rho)$ and $H_m^{(2)}(\beta_p\rho)$ in the pth layer. In view of the equation

$$i\epsilon_i\omega E_\phi=iK_p\frac{\partial H_z}{\rho\partial\phi}-M_p\frac{\partial H_z}{\partial\rho}, \tag{68}$$

it follows that

$$K_{m,p}^{+}=i\left[\frac{mK_p}{k\rho}-\left(\frac{M_p}{N_p}\right)^{1/2}\frac{J_m'(\beta_p\rho)}{J_m(\beta_p\rho)}\right] \tag{69}$$

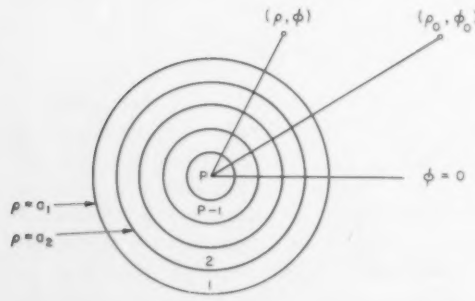

FIGURE 4. *A line source in the presence of a cylindrically stratified plasma column.*

and

$$K^-_{m,p}=-i\left[\frac{mK_p}{k\rho}-\left(\frac{M_p}{N_p}\right)^{1/2}\frac{H^{(2)\prime}_m(\beta_p\rho)}{H^{(2)}_m(\beta_p\rho)}\right]. \tag{70}$$

The index p ranges from 1 to P to signify the appropriate concentric region in the plasma.

The reflection coefficients at the interface between the pth and the $(p-1)$th layer are then

$$R_{m,p-1}=\frac{H^-_{z,m}}{H^+_{z,m}}=\frac{K^+_{m,p-1}-Z_{m,p}}{K^-_{m,p-1}+Z_{m,p}} \tag{71}$$

and

$$r_{m,p-1}=\frac{E^-_{\phi,m}}{E^+_{\phi,m}}=\frac{1/K^+_{m,p-1}-1/Z_{m,p}}{1/K^-_{m,p-1}+1/Z_{m,p}}, \tag{72}$$

where $Z_{m,p}$ is the impedance at the interface between the pth and the $(p-1)$th layer. $Z_{m,p}$ may be expressed in terms of the reflection coefficients $R_{m,p}$ and $r_{m,p}$ by again making use of Schelkunoff's nonuniform transmission line theory [13]. Thus,

$$Z_{m,p}=K^+_{m,p}\,\frac{1+r_{m,p}\chi^+_{m,e}(a_p,a_{p+1})\chi^-_{m,e}(a_{p+1},a_p)}{1+R_{m,p}\chi^+_{m,h}(a_p,a_{p+1})\chi^-_{m,h}(a_{p+1},a_p)}, \tag{73}$$

$$\chi^+_{m,e}(a_p,a_{p+1})=\frac{E^+_{\phi,m}(a_{p+1})}{E^+_{\phi,m}(a_p)}, \tag{74}$$

$$\chi^-_{m,e}(a_{p+1},a_p)=\frac{E^-_{\phi,m}(a_p)}{E^-_{\phi,m}(a_{p+1})}, \tag{75}$$

$$\chi^+_{m,h}(a_p,a_{p+1})=\frac{H^+_{z,m}(a_{p+1})}{H^+_{z,m}(a_p)}, \tag{76}$$

$$\chi^-_{m,h}(a_{p+1},a_p)=\frac{H^-_{z,m}(a_p)}{H^-_{z,m}(a_{p+1})}. \tag{77}$$

The χ's are clearly transmission factors which describe the fractional change of a wave as it propagates from one interface to the other within a layer. The numerator in eq (73) is thus a measure of the electric field at the pth interface taking into account the transmission through the layer, reflection at the $(p+1)$th layer and transmission back again to the pth layer. The denominator of eq (73) corresponds, in a similar manner, to the magnetic field.

The specific form of the transmission factors is

$$\chi^+_{m,e}(a_p,a_{p+1})=\frac{M_p\beta_pJ'_m(\beta_pa_{p+1})-mK_pJ_m(\beta_pa_{p+1})/a_{p+1}}{M_p\beta_pJ'_m(\beta_pa_p)-mK_pJ_m(\beta_pa_p)/a_p}, \tag{78}$$

$$\chi^-_{m,e}(a_{p+1},a_p)=\frac{M_p\beta_pH^{(2)\prime}_m(\beta_pa_p)-mK_pH^{(2)}_m(\beta_pa_p)/a_p}{M_p\beta_pH^{(2)\prime}_m(\beta_pa_{p+1})-mK_pH^{(2)}_m(\beta_pa_{p+1})/a_{p+1}}, \tag{79}$$

$$\chi^+_{m,h}(a_p,a_{p+1})=\frac{J_m(\beta_pa_{p+1})}{J_m(\beta_pa_p)}, \tag{80}$$

and

$$\chi^-_{m,h}(a_{p+1},a_p)=\frac{H^{(2)}_m(\beta_pa_p)}{H^{(2)}_m(\beta_pa_{p+1})}. \tag{81}$$

Then starting with the reflection coefficients $R_{m,P-1}$ and $r_{m,P-1}$, given by

$$R_{m,P-1}=\frac{K^+_{m,P-1}-K^+_{m,P}}{K^+_{m,P-1}+K^+_{m,P}}\qquad\text{and}\qquad r_{m,P-1}=\frac{1/K^+_{m,P-1}-1/K^+_{m,P}}{1/K^+_{m,P-1}+1/K^+_{m,P}}, \tag{82}$$

we may obtain $R_{m,P-2}$ and $r_{m,P-2}$ which in turn enables us to obtain $R_{m,P-3}$ and $r_{m,P-3}$. The process is continued until the outermost interface is reached, whence

$$R_{m,0}=\frac{K^+_{m,0}-Z_{m,1}}{K^-_{m,0}+Z_{m,1}} \qquad \text{and} \qquad r_{m,0}=\frac{1/K^+_{m,0}-1/Z_{m,1}}{1/K^-_{m,0}+1/Z_{m,1}}, \tag{83}$$

where

$$K^+_{m,0}=-i\,\frac{J'_m(k\rho)}{J_m(k\rho)}\,\eta_0 \tag{84}$$

and

$$K^-_{m,0}=i\,\frac{H^{(2)\prime}_m(k\rho)}{H^{(2)}_m(k\rho)}\,\eta_0 \tag{85}$$

If the scattered field H^s_z is written in the form

$$H^s_z=\frac{\epsilon_0\omega I}{4}\sum_{m=-\infty}^{+\infty} B_m H^{(2)}_m(k\rho_0)H^{(2)}_m(k\rho)e^{-im(\phi-\phi_0)}, \tag{86}$$

it follows that

$$B_m=-R_{m,0}\,\frac{J_m(ka)}{H^{(2)}_m(ka)}. \tag{87}$$

It may be readily verified that $R_{m,0}$ for the special case of a homogeneous plasma column (i. e., set $a_2=0$) is identical to the square bracket term on the right-hand side of eq (66).

For purposes of computation it is convenient to locate the line source at a great distance from the cylindrical column in terms of wavelength. The Hankel functions of argument $k\hat{\rho}$ and $k\rho_0$ may then be replaced by the first term of their asymptotic expansions since $k\hat{\rho}$ and $k\rho_0>>1$. Thus,

$$H^{\text{inc}}_z\cong\frac{\epsilon_0\omega I}{4}\left(\frac{2i}{\pi k\hat{\rho}}\right)^{1/2}e^{-ik\hat{\rho}} \tag{88}$$

and

$$H^s_z\cong\frac{\epsilon_0\omega I}{4}\left(\frac{2i}{\pi k\rho_0}\right)^{1/2}e^{-ik\rho_0}\sum_{m=-\infty}^{+\infty} B_m e^{im\pi/2}H^{(2)}_m(k\rho)e^{-im(\phi-\phi_0)}. \tag{89}$$

This form of the solution would have been obtained directly if plane wave incidence was assumed at the outset. It should be noted that the factor preceding the summation in eq (89) is just the value of H^{inc}_z evaluated at the center of the cylinder.

8. A Note on the Other Polarization

The results derived in the preceding sections are valid when the magnetic vector of the incident wave is parallel to the cylinder. The derivation for the other polarization, namely, that when the electric vector is parallel to the cylinder, is relatively trivial since the constant magnetic field H_0 has no influence. The dielectric constant is now simply a scalar and is equal to ϵ''. The explicit results for E-parallel polarization may be obtained from those of H-parallel polarization by making the following transforms:

$$\begin{aligned}
&H_z\rightarrow E_z\\
&E_z\rightarrow -H_z\\
&E_\phi\rightarrow -H_\phi\\
&E_y\rightarrow -H_y\\
&E_\rho\rightarrow -H_\rho\\
&M\rightarrow \ \ N\\
&N\rightarrow \ \ M
\end{aligned}$$

Also, of course, $K=0$ and now $M=(\epsilon_0/\epsilon'')$.

9. Appendix

In this appendix a rather approximate theory of electric currents in a partially ionized gas is given. The gas is supposed to consist of electrons of mass m_e and of charge $-\epsilon$; and positive ions of mass m_i and charge ϵ. Since the gas must be, to a close approximation, electrically neutral, the number of electrons and positive ions per unit volume is the same; each is taken equal to N_0. There are also N_a neutral atoms or molecules per unit volume with mass M_i indistinguishable from the positive ions. The electrons, positive ions and neutral atoms are regarded as three gases moving independently. The interaction between the electron and the ion gases is supposed to be smoothed into a continuous force. In fact only those forces are considered which result from macroscopic electromagnetic effects on charge and from friction-like action and reaction between the free charges and the uncharged background. Only mean velocities and forces are employed, and non-linearities are avoided by a perturbation treatment.

Further assumptions, adopted here, are that the velocity of the gas as a whole is zero, and that the gradients of the electron and ion pressures are zero. The removal of these latter restrictions would require that the problem be treated on the basis of magneto hydrodynamics [8, 16]. Such an approach has been given by Spitzer [7] for a wholly ionized gas and Cowling for a partially ionized gas [17].

The mean velocities of the electron gas, positive ion gas, and neutral gas are denoted by $\vec{v}_e$, $\vec{v}_i$, and $\vec{v}_a$, respectively. Since $m_e \ll m_i$, the electrons lose, on the average, a quantity of momentum equal to their mean momentum $m_e(\vec{v}_e - \vec{v}_i)$ at each collision with the charged ions. If the mean time between successive collisions is ν_0^{-1}, then the momentum lost by electrons in collision with positive ions per unit volume is $N_0 m_e(\vec{v}_e - \vec{v}_i)\nu_0$. Similarly, the momentum lost by electrons in collision with neutral atoms per unit volume is $N_0 m_e(\vec{v}_e - \vec{v}_a)\nu_e$ where ν_e^{-1} is the mean time between successive collisions of electrons with neutral atoms. Remembering that the momentum of the mass as a whole is zero, and since the mass of the electrons is negligible, it follows that

$$N_0 m_i \vec{v}_i + N_a m_i \vec{v}_a = 0, \tag{90}$$

and thus $\vec{v}_a = -\alpha \vec{v}_i$ where $\alpha = N_0/N_a$.

Now the electromagnetic forces acting on an electron are $-\epsilon(\vec{E} + \mu_0 \vec{v}_e \times \vec{H}_0)$ where $\vec{E}$ is the electric field of the wave and $\vec{H}_0$ is the constant magnetic field superimposed on the system. (It is assumed that $\vec{H}_0$ is much greater than the magnetic field $\vec{H}$ associated with the wave.) The equation of motion for the electron gas may then be written

$$m_e d\vec{v}_e/dt = m_e i\omega \vec{v}_e = -\epsilon[\vec{E} + \mu_0 \vec{v}_e \times \vec{H}_0] - m_e \nu_0(\vec{v}_e - \vec{v}_i) - m_e \nu_e(\vec{v}_e + \alpha \vec{v}_i). \tag{91}$$

The equation of motion for the positive ions can be obtained in an equally simple fashion. In this case, the momentum lost by the positive ions in collision with the electrons is

$$-N_0 m_e(\vec{v}_e - \vec{v}_i)\nu_0,$$

being equal and opposite to the quantity appearing above. The positive ions, in collision with the neutral atoms of the same mass, lose half their momentum relative to the neutral gas. Specifically, the momentum lost is $\frac{1}{2}N_0 m_i v_i(1+\alpha)\nu_i$ where ν_i^{-1} is the mean time between successive collisions of positive ions with neutral atoms. Therefore, the desired equation is

$$m_i d\vec{v}_i/dt = m_i i\omega \vec{v}_i = e[\vec{E} + \mu_0 \vec{v}_i \times \vec{H}_0] + m_e \nu_0(\vec{v}_e - \vec{v}_i) - m_i \nu_i \vec{v}_i(1+\alpha)/2. \tag{92}$$

Equations (91) and (92) may be solved for $\vec{v}_e$ and $\vec{v}_i$ in terms of $\vec{E}$ and other known quantities. The current density resulting from motion of the electrons and the ions is then equal to $N_0 e(\vec{v}_i - \vec{v}_e)$. Noting that the displacement current is $i\epsilon_0 \omega \vec{E}$ and the total current is $i(\epsilon)\omega \vec{E}$, it is seen that the dielectric tensor (ϵ) can be obtained from the relation

$$i(\epsilon)\omega\vec{E}=i\epsilon_0\omega\vec{E}+N_0e(\vec{v}_i-\vec{v}_e). \tag{93}$$

Choosing $\vec{H}_0$ to be directed along the z-axis, the dielectric tensor is found to have the form

$$(\epsilon)=\begin{bmatrix}\epsilon' & -iq & 0\\ iq & \epsilon' & 0\\ 0 & 0 & \epsilon''\end{bmatrix}. \tag{94}$$

The quantities ϵ', ϵ'', and q are functions of m_e, m_i, ν_e, ν_i, ν, N_0, N_a, and H_0. Explicit results for certain special cases are given in the body of the paper. To simplify the notation there, the results are expressed in terms of the positive real quantities ω_0 and ω_T which are defined by

$$\omega_0^2=N_0e^2/\epsilon_0m_e \tag{95}$$

and

$$\omega_T=-\mu_0H_0e/m_e \tag{96}$$

Also, in discussing certain special cases in the text, the subscript on ν is dropped.

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The solution given in these papers is a special case of eq (61) when the source is allowed to recede to infinity (i.e., $\rho_0\to\infty$).

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A New Decomposition Formula in the Theory of Elasticity[1]

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In this paper a new representation formula for the spherical components of displacement in the theory of elasticity is obtained. This formula may be used to reduce mixed boundary value problems for the elastic sphere to standard problems in potential theory.

1. Introduction

In treating boundary value problems in the theory of elasticity it is frequently convenient to represent the components of the displacement vector in terms of harmonic functions, and thus to reduce the elasticity problem to one in potential theory. Several such decompositions in terms of harmonic functions are known, [7, p. 342–343],[3] [1, p. 231], [4], the most widely used, perhaps, being the Papkovitch-Neuber decomposition (see for instance [7, p. 331]).

Half-space problems have been quite thoroughly treated from this point of view, as in fact have the first and second boundary value problems for the sphere. However, in treating mixed boundary value problems for the sphere, the standard decompositions are not suitable. Such mixed problems could be simplified considerably if it were possible to obtain convenient representations for the spherical components of displacement in terms of harmonic functions. We obtain such a representation in this paper and demonstrate how it yields in a simple manner solutions to standard mixed boundary value problems for the sphere.

In this paper whenever an index is repeated in a single term, summation from 1 to 3 will be implied. Use of the symbol $f_{,i}$ to denote $\partial f/\partial x_i$, where x_i is one of the rectangular coordinates (x_1, x_2, x_3), will be made throughout.

Let D be a three dimensional region which contains an open subset Q of a spherical surface of radius a with center at the origin of a spherical coordinate system (r, θ, φ). The region D is assumed to be a portion of an isotropic homogeneous elastic medium.

We suppose that D does not contain the origin. Furthermore, we assume that every ray from the origin which intersects D, intersects it in a single line segment.

In D the component of the displacement vector satisfies the equilibrium equations

$$u_{i,jj}+\alpha u_{j,ji}=0 \tag{1.1}$$

where α involves the elastic constants ($\alpha=(1-2\sigma)^{-1}$; where σ is Poisson's ratio). As far as our representation formulas are concerned we need put no restrictions on the constant α other than that it be different from -1. However, all physically interesting values of α lie in the interval $\frac{1}{3}<\alpha<\infty$.

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[3] Figures in brackets indicate the literature references at the end of this paper.

In this paper we obtain the following representation for the spherical components of displacement in D:

$$x_j u_j = r^2 \left\{ 2G + (2-\alpha) r \partial G/\partial r - \alpha r \frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r}\right)\right\} + r \frac{\partial \psi}{\partial r} \tag{1.2}$$

$$\epsilon_{ijk} x_j u_k = r^2 \left\{ -(2+3\alpha)\epsilon_{ijk} x_j G_{,k} - \alpha r \frac{\partial}{\partial r} \epsilon_{ijk} x_j G_{,k} - \left(H + r \frac{\partial H}{\partial r}\right)_{,i} \right\}$$

$$+ \left\{ x_i \left[r \frac{\partial H}{\partial r} + 2r \frac{\partial}{\partial r}\left(r \frac{\partial H}{\partial r}\right)\right] - r^2 \left[r \frac{\partial H}{\partial r}\right]_{,i} \right\} + \epsilon_{ijk} x_j \psi_{,k} \tag{1.3}$$

where ψ, G, and H are harmonic functions in D. Consequently all terms in braces in (1.2) and (1.3) are also harmonic. In (1.3) ϵ_{ijk} is the permutation symbol

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are equal} \\ 1 & \text{if } ijk \text{ is an even permutation of } 1\ 2\ 3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1\ 2\ 3. \end{cases} \tag{1.4}$$

2. Representation Formulas

We first rewrite (1.1) as

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} u_{l,m} + (\alpha+1)\, u_{j,j} = 0. \tag{2.1}$$

For $\alpha \neq -1$ we see from (1.1) that $u_{j,j}$ is harmonic. It has been shown that the representation

$$u_{j,j} = \left(1 + r \frac{\partial}{\partial r}\right)\left(6G + 4r \frac{\partial G}{\partial r}\right), \tag{2.2}$$

where G is harmonic, is valid in D (see for instance Bramble [2]). Equation (2.1) then may be integrated to give

$$\epsilon_{ijk}\, u_{j,k} = (\alpha+1)\, \epsilon_{ijk}\, x_j \left(6G + 4r \frac{\partial G}{\partial r}\right)_{,k} + \chi_{,i} \tag{2.3}$$

where χ is a harmonic function. We express χ in D as

$$\chi = \Phi + r \frac{\partial \Phi}{\partial r}. \tag{2.4}$$

Thus u_j is the component of a vector whose divergence is represented by (2.2) and whose curl is expressible by (2.3). The component u_j may clearly be expressed as the sum of a particular solution of (2.2) and (2.3) and the gradient of a harmonic function, i.e.,

$$u_j = 2x_j \left[G + (\alpha+2)\, r \frac{\partial G}{\partial r}\right] - r^2 \left[2\,(1+2\alpha)\, G_{,j} + \alpha r \frac{\partial}{\partial r} G_{,j}\right] + \epsilon_{jik}\, x_i \Phi_{,k} + \psi_{,j} \tag{2.5}$$

where ψ is harmonic in D.

Using (2.5) we now form the expressions for $x_j u_j$ and $\epsilon_{ijk} x_j u_k$ which are respectively the scalar product and the vector product of the radius vector and the displacement vector.

$$x_j u_j = r^2 \left\{ 2G + (2-\alpha)\, r \frac{\partial G}{\partial r} - \alpha r \frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r}\right)\right\} + r \frac{\partial \psi}{\partial r} \tag{2.6}$$

and

$$\epsilon_{ijk} x_j u_k = r^2 \left\{ -(2+3\alpha)\, \epsilon_{ijk} x_j G_{,k} - \alpha r \frac{\partial}{\partial r} \epsilon_{ijk} x_j G_{,k} - \Phi_i \right\} + x_i r \frac{\partial \Phi}{\partial r} + \epsilon_{ijk} x_j \psi_{,k}. \tag{2.7}$$

Equation (2.6) is precisely the desired expression (1.2). In order to obtain (1.3) we substitute for Φ in D (see Bramble [2]),

$$\Phi = H + 2r\frac{\partial H}{\partial r} \tag{2.8}$$

where H is harmonic. We then write

$$x_i r\frac{\partial \Phi}{\partial r} = \left[x_i \left\{ r\frac{\partial H}{\partial r} + 2r\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right)\right\} - r^2\left(r\frac{\partial H}{\partial r}\right)_{,i}\right] + r^2\left(r\frac{\partial H}{\partial r}\right)_{,i}. \tag{2.9}$$

As noted previously, the term in square brackets is harmonic. Equations (2.7), (2.8) and (2.9) yield immediately the desired representation (1.3).

In obtaining our decompositions we have frequently made use of the fact that in the region in question any harmonic function could be represented as $\left(1+a_1 r\frac{\partial}{\partial r}\right)\left(1+a_2 r\frac{\partial}{\partial r}\right) g$ where g is harmonic. We have used only representations in which a_1 and a_2 were non-negative. Bramble [2] has shown that such a representation is valid in D regardless of the sign of a_1 and a_2. However, for a_1 and a_2 non-negative it is easily verified that such a representation is also valid in any region whose boundery is star-shaped with respect to the origin. Consequently (1.2) and (1.3) are valid in any such star shaped region. In the next section we use the representations (1.2) and (1.3) to solve certain boundary value problems for the elastic sphere.

If the elastic region is exterior to a surface C which is star-shaped with respect to the origin and if the components of the displacement vector are $\mathbf{O}(r^{-2})$ as $r\to\infty$, it can again be shown that the decompositions (1.2) and (1.3) are valid therein.

3. Mixed Boundary Value Problems for the Sphere

Let u_i satisfy (1.1) on the interior of an elastic sphere. On the surface $r=a$ let

$$\begin{aligned} &\mu^{-1}a_1\,\tau_{ij}x_i x_j + a_2 x_j u_j = f(\theta,\varphi) \\ &\mu^{-1}b_1\,\epsilon_{ijk}\tau_{kl}x_j x_l + b_2\epsilon_{ijk}x_j u_k = g_i(\theta,\varphi). \end{aligned} \tag{3.1}$$

where τ_{ij} is the stress tensor, μ is the shear modulus, and where f and g_i are prescribed boundary data. Equations (3.1) may be written equivalently as

$$\begin{aligned} &a_1(\alpha+1)r\frac{\partial}{\partial r}(x_j u_j) + [a_1(\alpha-3)+a_2]x_j u_j + a_1(\alpha-1)\epsilon_{ijk}x_i\frac{\partial}{\partial x_j}\epsilon_{klm}x_l u_m = f(\theta,\varphi) \\ &b_1 r\frac{\partial}{\partial r}\epsilon_{ijk}x_j u_k + (b_2-2b_1)\epsilon_{ijk}x_j u_k + b_1\epsilon_{ijk}x_j(x_l u_l)_{,k} = g_i(\theta,\varphi) \end{aligned} \tag{3.2}$$

on $r=a$. Using (1.2) and (1.3) we can actually obtain the following representation for the left hand side of (3.1) in terms of harmonic functions

$$\begin{aligned} \mu^{-1}a_1\tau_{ij}x_i x_j + a_2 x_j u_j = r^2&\left\{ -2a_1\alpha\left(r\frac{\partial}{\partial r}\right)^3 G - a_2\alpha\left(r\frac{\partial}{\partial r}\right)^2 G + [2a_1(4\alpha-1)+a_2(2-\alpha)]\, r\frac{\partial G}{\partial r}\right. \\ &\left. + 2[(3\alpha-1)a_1+a_2]G\right\} + 2a_1 r\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - (2a_1-a_2)r\frac{\partial \psi}{\partial r} \\ \mu^{-1}b_1\epsilon_{ijk}\tau_{kl}x_j x_l + b_2\epsilon_{ijk}x_j u_k = r^2&\epsilon_{ijk}x_j\frac{\partial}{\partial x_k}\left\{ -2\alpha b_1\left(r\frac{\partial}{\partial r}\right)^2 G - (b_2+4b_1)\alpha r\frac{\partial G}{\partial r} + [2b_1-(2+3\alpha)b_2]G\right\} \\ &-2r^2\left(b_1 r\frac{\partial}{\partial r}+b_2\right)\left(H + r\frac{\partial H}{\partial r}\right)_{,i} + \epsilon_{ijk}x_j\frac{\partial}{\partial x_k}\left\{ 2b_1 r\frac{\partial \psi}{\partial r} + (b_2-2b_1)\psi\right\} \\ &+\left(b_1 r\frac{\partial}{\partial r}+b_2-2b_1\right)\left\{ x_i\left[r\frac{\partial H}{\partial r} + 2r\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right)\right] - r^2\left(r\frac{\partial H}{\partial r}\right)_{,i}\right\}. \end{aligned} \tag{3.3}$$

From (3.3) we obtain expressions of the form

$$A_1\left(r\frac{\partial}{\partial r}\right)^3 G+A_2\left(r\frac{\partial}{\partial r}\right)^2 G+A_3 r\frac{\partial G}{\partial r}+A_4 G+A_5 r\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+A_6 r\frac{\partial \psi}{\partial r}=P(f),$$

$$\epsilon_{ijk}x_j\frac{\partial}{\partial x_k}\left\{B_1\left(r\frac{\partial}{\partial r}\right)^2 G+B_2 r\frac{\partial G}{\partial r}+B_3 G+B_4 r\frac{\partial \psi}{\partial r}+B_5\psi\right\} \quad (3.4)$$

$$+\left\{x_i\left(2r\frac{\partial}{\partial r}+1\right)\left(r\frac{\partial}{\partial r}\right)\left(B_6 r\frac{\partial H}{\partial r}+B_7 H\right)-r^2\left[r\frac{\partial}{\partial r}\left(B_6 r\frac{\partial H}{\partial r}+B_7 H\right)\right]_{,i}\right\}$$

$$-a^2\left[\left(1+r\frac{\partial}{\partial r}\right)\left(B_6 r\frac{\partial H}{\partial r}+B_7 H\right)\right]_{,i}=P(g_i)$$

valid throughout D. The constants A_i and B_i are determined immediately from (3.3) and $P(f)$ and $P(g_i)$ denote Poisson integrals.

Forming $x_i P(g_i)$ we obtain the expression

$$B_6\left\{\left(r\frac{\partial}{\partial r}\right)^3 H+\left(r\frac{\partial}{\partial r}\right)^2 H\right\}+B_7\left\{\left(r\frac{\partial}{\partial r}\right)^2 H+r\frac{\partial H}{\partial r}\right\}=\frac{x_i P(g_i)}{r^2-a^2}. \quad (3.5)$$

Under suitable conditions on B_6 and B_7 this expression may be integrated to give an expression for H. With H known throughout D, equations (3.4) give two identities in the two unknowns G and ψ. The function ψ can be eliminated by taking the proper combination of r derivatives of the first of equations (3.4) and r derivatives of the expression formed by operating on the second of equations (3.4) with $\epsilon_{lmi}x_l\frac{\partial}{\partial x_m}$. This results in an equation involving a linear combination of r derivatives of G. If this equation can be solved for G then the insertion of G into the first of equations (3.4) yields the expression for ψ which completely determines the displacement vector in D. It is a simple matter to obtain a formal solution for G, for we have an equation of the form

$$\sum_{i=0}^{N} a_i\left(r\frac{\partial}{\partial r}\right)^i G=F. \quad (3.6)$$

We expand the known harmonic function F in terms of spherical harmonics, i.e.,

$$\sum_{i=0}^{N} a_i\left(r\frac{\partial}{\partial r}\right)^i G=\sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n P_n^m(\cos\theta)[A_{nm}\cos m\phi+B_{nm}\sin m\phi]. \quad (3.7)$$

The solution to this equation is

$$G=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\sum_{i=0}^{N} a_i n^i\right]^{-1} r^n P_n^m(\cos\theta)\,[A_{nm}\cos m\phi+B_{nm}\sin m\phi] \quad (3.8)$$

provided

$$\sum_{i=0}^{N} a_i n^i\neq 0 \quad (3.9)$$

for $n=0, 1, 2, \ldots$, or any positive integer. For a given α, (3.9) imposes certain restrictions on the constants a_1, a_2, b_1, and b_2.

As a particular example consider the case in which a_2 and b_1 in (3.1) are zero. The constants a_1 and b_2 can without loss be taken as unity. Thus equation (3.4) becomes

$$a^2\left[-2\alpha\left(r\frac{\partial}{\partial r}\right)^3 G+2(4\alpha+1)\,r\frac{\partial G}{\partial r}+2(3\alpha-1)\,G\right]+2r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}-1\right)\psi=P(f) \quad (3.10)$$

and

$$a^2\epsilon_{ijk}x_j\frac{\partial}{\partial x_k}\left[-\alpha r\frac{\partial G}{\partial r}-(2+3\alpha)G\right]-\epsilon_{ijk}x_j\psi_{,k}-a^2\left(H+r\frac{\partial H}{\partial r}\right)_{,i}$$
$$+\left\{x_i\left[r\frac{\partial H}{\partial r}+2\left(r\frac{\partial}{\partial r}\right)^2H\right]-r^2\left(r\frac{\partial H}{\partial r}\right)_{,i}\right\}=P(g_i) \quad (3.11)$$

where $P(g_i)$ and $P(f)$ denote Poisson's integrals. If we take the scalar product of the radius vector and (3.11) we obtain the differential equation for H

$$r\frac{\partial}{\partial r}\left[H+r\frac{\partial H}{\partial r}\right]=\frac{x_iP(g_i)}{r^2-a^2} \quad (3.12)$$

which yields the result

$$H=\int_0^r\rho^{-1}\frac{(r-\rho)x_iP(g_i)}{(\rho^2-a^2)}d\rho. \quad (3.13)$$

It is possible to add an arbitrary constant to the right hand side of (3.13) but since only derivatives of H occur in (1.3) we may take the constant equal to zero. We now operate on (3.11)

with the operator $\epsilon_{lmi}x_l\dfrac{\partial}{\partial x_m}$ to obtain the expression

$$r\frac{\partial^2}{\partial r^2}\left\{r\left[\psi-(2+3\alpha)a^2G-a^2\alpha r\frac{\partial G}{\partial r}\right]\right\}=\epsilon_{ijk}x_i\frac{\partial}{\partial x_j}P(g_k) \quad (3.14)$$

which yields for ψ

$$\psi=a^2\left[(2+3\alpha)G+\alpha r\frac{\partial G}{\partial r}\right]-r^{-1}\int_0^r\rho^{-1}(r-\rho)\epsilon_{ijk}x_i\frac{\partial}{\partial x_j}P(g_k)d\rho. \quad (3.15)$$

Again we have set the arbitrary additive constant equal to zero in (3.15) since only derivatives of ψ occur in (1.2) and (1.3). Substitution of (3.15) into (3.10) yields the result

$$\left(r\frac{\partial}{\partial r}-c_1\right)\left(r\frac{\partial}{\partial r}-c_2\right)G=[4a^2(\alpha+1)]^{-1}R \quad (3.16)$$

where

$$\begin{aligned}c_1&=\{(3-\alpha)+[17-22\alpha-23\alpha^2]^{1/2}\}[4(\alpha+1)]^{-1}\\ c_2&=\{(3-\alpha)-[17-22\alpha-23\alpha^2]^{1/2}\}[4(\alpha+1)]^{-1}\end{aligned} \quad (3.17)$$

and

$$R=P(f)+2\left(r\frac{\partial}{\partial r}-1\right)\left[r^{-1}\int_0^r\epsilon_{ijk}x_i\frac{\partial}{\partial x_j}P(g_k)d\rho\right]. \quad (3.18)$$

For $\alpha>1/3$ (which corresponds to the range of σ which is of physical interest, i.e., $-1<\sigma<1/2$) c_1 and c_2 have real part less than unity. Thus if we operate on (3.16) by $r\partial/\partial r$ we obtain

$$\left(r\frac{\partial}{\partial r}-c_1\right)\left(r\frac{\partial}{\partial r}-c_2\right)r\frac{\partial G}{\partial r}=[4a^2(\alpha+1)]^{-1}\,r\frac{\partial R}{\partial r} \quad (3.19)$$

an expression which can easily be integrated for G, giving

$$G=[4a^2(\alpha+1)]^{-1}\left\{\frac{R(0,\theta,\phi)}{c_1c_2}+\int_0^r\rho^{c_2-1}\int_0^\rho\rho_1^{c_1-c_2-1}\int_0^{\rho_1}\rho_2^{-c_1}\frac{\partial R}{\partial\rho_2}\,d\rho_2d\rho_1d\rho\right\}. \quad (3.20)$$

If the indicated integration is performed we obtain for $c_1 \neq c_2$ the expression

$$G=[4a^2(\alpha+1)]^{-1}\left\{\frac{R(r,\theta,\phi)}{c_1c_2}+(c_1-c_2)^{-1}\left[\frac{r^{c_1}}{c_1}\int_0^r \rho^{-c_1}\frac{\partial R}{\partial \rho}\,d\rho-\frac{r^{c_2}}{c_2}\int_0^r \rho^{-c_2}\frac{\partial R}{\partial \rho}\,d\rho\right]\right\}. \quad (3.21)$$

If $c_1=c_2$ (which means that α satisfies the equation $17-22\alpha-23\alpha^2=0$) (3.20) yields upon integration

$$G=[4a^2c_1^2(\alpha+1)]^{-1}\left\{c_1r^{c_1}\int_0^r \rho^{-c_1}\log(r/\rho)\,\frac{\partial R}{\partial \rho}\,d\rho-r^{c_1}\int_0^r \rho^{-c_1}\frac{\partial R}{\partial \rho}\,d\rho+R\right\}. \quad (3.22)$$

The function ψ is now determined by insertion of (3.21) or (3.22) in (3.15), and the problem is solved.

4. Concluding Remarks

In a recent paper [3], the authors indicated how to obtain an explicit continuation across a spherical surface Q on which f and g_i of (3.1) were zero. The decomposition (1.2), (1.3) could also be used to obtain this continuation formula. It is also possible to obtain in a straightforward way from (1.2) and (1.3) sphere theorems with various combinations of surface tractions and displacements vanishing on the surface of the sphere.

If in (1.2) and (1.3) α is allowed to tend to infinity while at the same time αG tends to a harmonic function F, then these decompositions may be interpreted as representation formulas for the spherical components of the velocity vector in the slow motion of viscous fluids. (See for instance [6, p. 522].)

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Pointwise Bounds in the Cauchy Problem of Elastic Plates

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Let an elastic plate occupy a region D with boundary C. On a portion Σ of C the deflection, slope, moment, and shear are measured. In terms of these data, and crude bounds for the maximum deflection and moment, pointwise bounds for the value of the deflection and its derivatives in D are obtained. These bounds are such that if the data on Σ and the loading perpendicular to the plane of the plate tend to zero, the deflection tends to zero at every point in D. Similar bounds are obtained in the slow flow problem for a viscous fluid.

1. Introduction

The following plate problem is considered. Let an elastic plate occupying a region D in the xy-plane be subjected to a load $p(x,y)$ normal to the plane of the plate. The deflection v satisfies the equation

$$\Delta^2 v=\left(\frac{\partial^4 v}{\partial x^4}+2\,\frac{\partial^4 v}{\partial x^2 dy^2}+\frac{\partial^4 v}{\partial y^4}\right)=p/\mathscr{D}, \tag{1.1}$$

where D denotes the plate rigidity (see [3, p. 488]).[1] We assume that on a portion Σ of the boundary of D we are able to measure the deflection, slope, the sum of the two principle moments, and the normal component of shear with known error; or equivalently we assume

$$\left.\begin{aligned} &\int_\Sigma (v-f)^2 d\sigma\leq\alpha_1, \qquad \int_\Sigma (v_{,i}-F_i)(v_{,i}-F_i)d\sigma\leq\alpha_2\\ &\int_\Sigma (\Delta v-g)^2 d\sigma\leq\alpha_3, \qquad \int_\Sigma (\Delta v_{,i}-G_i)(\Delta v_{,i}-G_i)d\sigma\leq\alpha_4 \end{aligned}\right\} \tag{1.2}$$

where f, F_i, g, and G_i are the measured values of v,$v_{,i}$, Δv, and $\Delta v_{,i}$ respectively. In (1.2) the comma denotes differentiation and a repeated subscript i implies summation over i ($i=1, 2$). The Δ symbol is used throughout to denote the Laplace operator.

In (1.2) then the α_i are assumed to be small known constants. We wish to obtain upper and lower bounds for the solution v at any point in R.

It is well known that as it stands the Cauchy problem for the biharmonic equation is not well posed (see Hadamard [1]). The solution does not depend in a continuous way on the boundary data. We leave aside the question of existence of solution and merely remark that in most physical situations in which one would be interested in considering such a problem the question does not arise. For instance, we might wish to treat an ordinary plate boundary value problem, but find that part of the boundary is inaccessible for measurement of boundary data whereas on another portion of the boundary it is possible to measure v, $\dfrac{\partial v}{\partial n}$, Δv, and $\dfrac{\partial}{\partial n}(\Delta v)$.

If in addition to the Cauchy data we have uniform bounds for both v and Δv throughout D then the solution to the plate problem will be seen to be stable. This is in fact an immediate consequence of the results of Laurentiev [2].

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[1] Figures in brackets indicate the literature references at the end of this paper.

In order to simplify the procedure somewhat we decompose the solution v into

$$v=v_p+v', \tag{1.3}$$

where v_p is any particular solution of (1.1) defined in D and v' satisfies

$$\Delta^2 v'=0 \tag{1.4}$$

in D. We approximate v' by a biharmonic function φ and set $w=v'-\varphi$. Thus w satisfies in D

$$\Delta^2 w=0. \tag{1.5}$$

Also bounds for the Cauchy expressions

$$\left.\begin{aligned} \eta_1&=\int_\Sigma w^2 d\sigma, & \eta_2&=\int_\Sigma w_{,t} w_{,t} d\sigma \\ \epsilon_1&=\int_\Sigma (\Delta w)^2 d\sigma, & \epsilon_2&=\int_\Sigma \Delta w_{,t} \Delta w_{,t} d\sigma \end{aligned}\right\} \tag{1.6}$$

are known. In fact

$$\eta_1=\int_\Sigma [(v'+v_p-f)-(\phi+v_p-f)]^2 d\sigma \leq 2\alpha_1+2\int_\Sigma (\phi+v_p-f)^2 d\sigma \tag{1.7}$$

with similar inequalities for η_2, ϵ_1, and ϵ_2. It is clear that if α_1 is small and ϕ can be chosen to approximate f-v_p in mean square on Σ then η_1 will be small. A similar statement holds for η_2, ϵ_1, and ϵ_2. We shall show in this paper that it is possible to obtain an *a priori* bound for w of the following type at any point in R:

$$|w(P)|^2 \leq A_1 M_1^{1-\delta_1}\{a_1\eta_1+a_2\eta_2\}^{\delta_1}+A_2 M_2^{1-\delta_2}\{b_1\epsilon_1+b_2\epsilon_2\}^{\delta_2}, \tag{1.8}$$

where A_1, A_2, a_1, a_2, b_1, b_2, are determined constants, δ_1 and δ_2 are constants satisfying

$$0<\delta_2<\delta_1<1, \tag{1.9}$$

and M_1 and M_2 are the uniform bounds for w, and Δw in D. It is clear that knowledge of uniform bounds for v' and $\Delta v'$ implies knowledge of uniform bounds for w and Δw since w-v' is a known function.

With an inequality of type (1.8) it is clear from (1.7) and the corresponding inequalities for η_2, ϵ_1, and ϵ_2 that if the α_i are sufficiently small and the Rayleigh-Ritz technique is used to make the integral expression on the extreme right small then we obtain close upper and lower bounds for $v'(P)$. This paper will be concerned then with the establishing of (1.8).

2. *A priori* Bounds

We consider a two-dimensional region D bounded by a closed surface C. On a portion Σ of C, Cauchy data are prescribed. We denote the remainder of the surface (C-Σ) by $\overline{\Sigma}$. (As will become apparent later the portion $\overline{\Sigma}$ need not actually be prescribed.) We now choose the origin of a polar coordinate system at a point P_0 which has the property that there exists a circle $K(r)$ of radius r and center at P_0 such that the intersection of C and $K(r)$ contains no point of $\overline{\Sigma}$. (If two or more disjoint regions are formed by the intersection of $K(r)$ and D, it is sufficient that the boundary of any one of these regions contains no points of $\overline{\Sigma}$.) We denote by D_r the intersection of $K(r)$ and D (the portion whose boundary contains no points of $\overline{\Sigma}$ if two or more disjoint regions are formed) and require that P_0 lie outside of D_r. We denote by R the radius of the largest circle satisfying the above conditions (for a fixed P_0) and by r_0 the radius of the smallest such circle. As indicated in the previous paper [5] we locate P_0 in such a

way that $K(r_0)$ is tangent to D at some point of Σ. Finally we denote by $\omega(r)$ the portion of the surface of $K(r)$ which lies interior to C.

As in [5] we introduce for any function u the notation

$$\left.\begin{aligned}\epsilon_1&=\int_\Sigma u^2 d\sigma\\ \epsilon_2&=\int_\Sigma\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]d\sigma,\end{aligned}\right\} \tag{2.1}$$

where $d\sigma$ denotes an element of surface area on Σ, and let

$$A_r(u)=\iint_{D_r}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy. \tag{2.2}$$

In [5] we proved that if u satisfies

$$\Delta u=0, \qquad |u|\leq M \text{ in } D, \tag{2.3}$$

then the functional

$$v(r,u)=\log\left[\int_{r_0}^{r}\rho^{-1}A_\rho(u)d\rho+\kappa_1\epsilon_1+\kappa_2\epsilon_2\right]+C(\log r)^2 \tag{2.4}$$

is a convex functional of $\log r$ in $r_0<r<R$, where

$$C=\frac{1}{2}(1+\sqrt{2})r_0R^{-1}; \qquad \kappa_1=\frac{3}{2}r_0^{-1}; \qquad \kappa_2=\frac{1}{2}\{r_0^2[\log(R/r_0)]^2+R^2\}r_0^{-1}. \tag{2.5}$$

It was emphasized in [5] that the constants in [2.5] are usually not the optimal constants, which in any specific problem depend on the geometry of Σ and D as well as the location of P_0.

An evaluation of the integral in (2.4) leads immediately to the inequality

$$\left|\int_{r_0}^{r}\rho^{-1}A_\rho(u)d\rho-\frac{1}{2}r^{-1}\int_{\omega(r)}u^2ds\right|\leq r_0^{-1}\epsilon_1+r_0\log(R/r_0)\epsilon_2. \tag{2.6}$$

Following the procedure indicated in [5] we are lead to the inequality

$$r^{-1}\int_{\omega(r)}u^2ds+r_0^{-1}\epsilon_1+R^2r_0^{-1}\epsilon_2\leq 2BM^{2(1-\delta)}(\kappa_1\epsilon_1+\kappa_2\epsilon_2)^\delta. \tag{2.7}$$

The constant B is easily computed and in this case

$$\delta=\log(R/r)/\log(R/r_0). \tag{2.8}$$

In the plate problem under consideration (1.5), (1.6) we now set

$$\Delta w=u. \tag{2.9}$$

Then (2.7) is valid with u replaced by Δw. Since we shall need it later we obtain now by integration the following inequality

$$\iint_{D_r}(\Delta w)^2dxdy+\frac{1}{2}(r^2-r_0^2)r_0^{-1}\epsilon_1+\frac{1}{2}(r^2-r_0^2)R^2r_0^{-1}\epsilon_2$$
$$\leq 2B\,\frac{[r^2M^{2(1-\delta)}(\kappa_1\epsilon_1+\kappa_2\epsilon_2)^\delta]\left[1-\dfrac{r_0^2}{r^2}\left(\dfrac{\kappa_1\epsilon_1+\kappa_2\epsilon_2}{M^2}\right)^{1-\delta}\right]\log(R/r_0)}{\log[M^2R^2/r_0^2(\kappa_1\epsilon_1+\kappa_2\epsilon_2)]}. \tag{2.10}$$

Note that for

$$\kappa_1\epsilon_1+\kappa_2\epsilon_2\leq M^2, \tag{2.11}$$

which must hold if the estimates are to be useful, we obtain the cruder but simpler inequality

$$\iint_{D_r}(\Delta w)^2 dsdy+\frac{1}{2}(r^2-r_0^2)^{-1}r_0\epsilon_1+\frac{1}{2}(r^2-r_0^2)R^2r_0^{-1}\epsilon_2\leq Br^2M^{2(1-\delta)}(\kappa_1\epsilon_1+\kappa_2\epsilon_2)^{\delta}. \tag{2.12}$$

We wish to find a bound for $\int_{\omega(r)} w^2 ds$. But once such a bound is known then pointwise bounds for w at points in D_r follow at once; for we need merely introduce the biharmonic Green's function $\Gamma(P,Q)$ which vanishes with its Laplacian on the circle of radius r and center at P_0, and has a fundamental singularity at a point P in D_r. We then make use of the identity

$$w(P)=-\int_{\omega(r)} w\frac{\partial}{\partial n}(\Delta\Gamma)ds-\int_{\omega(r)}\Delta w\frac{\partial\Gamma}{\partial n}ds-\int_{\Sigma'}\left[w\frac{\partial}{\partial n}(\Delta\Gamma)-\frac{\partial w}{\partial n}\Delta\Gamma+\Delta w\frac{\partial\Gamma}{\partial n}-\frac{\partial}{\partial n}(\Delta w)\Gamma\right]d\sigma, \tag{2.13}$$

where Σ' denotes the portion of Σ which lies interior to $K(r)$. An application of Schwarz's inequality then yields the desired result.

In order to obtain the bound for $\int_{\omega(r)} w^2 ds$ we introduce the following functional:

$$U(r,w)=\log\left\{\int_{r_0}^{r}\rho^{-p}\left[A_\rho(w)+\iint_{D_\rho}w\Delta w dxdy\right]d\rho+l_1\eta_1+l_2\eta_2+l_3\eta_3+l_4\eta_4\right\}+C_1r^{-2q}, \tag{2.14}$$

where

$$\left.\begin{aligned}&\eta_1=\int_\Sigma w^2 d\sigma, \qquad \eta_2=\int_\Sigma w_{,i}w_{,i}\,d\sigma,\\ &\eta_3=\iint_{D_{R_1}}(\Delta w)^2dxdy,\ \eta_4=2BR_1M^{2(1-\beta)}(\kappa_1\epsilon+\kappa_2\epsilon_2)^{\beta}\end{aligned}\right\} \tag{2.15}$$

for

$$r_0<r<R_1<R; \qquad \beta=\log(R/R_1)/\log(R/r_0) \tag{2.16}$$

and

$$p>1, \qquad q>p-1. \tag{2.17}$$

We demonstrate that for p and q satisfying (2.17) and the constants l_1, l_2, l_3, l_4, and C_1 suitably chosen, U is a convex function of r^{-q}.

To this end we introduce the notation

$$U(r,w)=\log E(r,w)+C_1r^{-2q} \tag{2.18}$$

(where E is the term in braces in (2.14)), and form

$$\begin{aligned}q^2E^2\frac{\partial^2U}{(\partial r^{-q})^2}=E\Big\{&(q+1-p)r^{2q+1-p}\left[A_r(w)+\iint_{D_r}w\Delta w dxdy\right]+r^{2(q+1)-p}\Big(\int_{\omega(r)}[|\operatorname{grad} w|^2\\ &+w\Delta w]ds\Big)\Big\}-r^{2(q+1-p)}\left[A_r(w)+\iint_{D_r}w\Delta w dxdy\right]^2+2q^2C_1E^2.\end{aligned} \tag{2.19}$$

As in [5] we obtain in a straightforward manner the inequality

$$m_1\eta_1+m_2\eta_2\leq E-\frac{1}{2}r^{-p}\int_{\omega(r)}w^2ds-\frac{1}{2}(p-1)\iint_{D_r}\rho^{-(p+1)}w^2dxdy-l_3\eta_3-l_4\eta_4\leq n_1\eta_1+n_2\eta_2, \tag{2.20}$$

where

$$\left.\begin{aligned} m_1 &= l_1 - \frac{(p+2)r_0^{-p}}{2(p+1)}, \qquad m_2 = l_2 - \frac{r_0^{-(p-2)}}{2(p+1)}, \\ n_1 &= l_1 + \frac{(p+2)r_0^{-p}}{2(p+1)}, \qquad n_2 = l_2 + \frac{r_0^{-(p-2)}}{2(p+1)}. \end{aligned}\right\} \tag{2.21}$$

The constants l_1 and l_2 are to be so chosen that m_1 and m_2 are positive.

By Green's theorem

$$A_r(w) + \iint_{D_r} w\Delta w\,dx\,dy = \int_{w(r)} w\,\frac{\partial w}{\partial r}\,ds + \int_{\Sigma'} w\,\frac{\partial w}{\partial n}\,d\sigma. \tag{2.22}$$

Thus by applying Schwarz's inequality in (2.22) and using the fact that the arithmetic mean is not less than the geometric mean we obtain from (2.19) using (2.21), (2.7), and (2.9)

$$\begin{aligned} q^2 E^2 \frac{\partial^2 U}{(\partial r^{-q})^2} \geq \frac{1}{2} r^{-p} \int_{\omega(r)} w^2 ds \Bigg\{ & (q+1-p) r^{2q+1-p} \left[A_r(w) - \frac{b_1}{2} r^{-2} \iint_{D_r} w^2 dx\,dy - \frac{r^2}{2b_1}\eta_3 \right] \\ & + r^{2(q+1)-p} \left[\int_{\omega(r)} \left\{ \left(\frac{\partial w}{\partial s}\right)^2 - \left(\frac{\partial w}{\partial r}\right)^2 \right\} ds - \frac{b_2 r^{-2}}{2} \int_{\omega(r)} w^2 ds - \frac{r^2}{2b_2}\eta_4 \right] - \frac{\alpha_1}{2}(\gamma_1\eta_1 + \gamma_2\eta_2) \Bigg\} \\ & + \left[\frac{(p-1)}{2} \iint_{D_r} r^{-(p+1)} w^2 dx\,dy + m_1\eta_1 + m_2\eta_2 + l_3\eta_3 + l_4\eta_4 \right] \\ & \times \Bigg\{ (q+1-p) r^{2q+1-p} \left[A_r(\omega) - \frac{b_3 r^{-2}}{2} \iint_{D_r} w^2 dx\,dy - \frac{r^2}{2b_3}\eta_3 \right] \\ & \qquad + r^{2(q+1)-p} \left[\int_{\omega(r)} |\mathrm{grad} w|^2 ds - \frac{b_4}{2} \int_{\omega(r)} w^2 ds - \frac{\eta_4}{2b_4} \right] \Bigg\} \\ & - \frac{1}{2\alpha_1} \int_{\omega(r)} \left(\frac{\partial u}{\partial r}\right)^2 ds (\gamma_1\eta_1 + \gamma_2\eta_2) - \eta_1\eta_2 \\ & + 2q^2 C_1 \left\{ \frac{r^{-p}}{2} \int_{\omega(r)} w^2 ds + \frac{(p-1)}{2} \iint_{D_r} r^{-(p+1)} w^2 dx\,dy + l_3\eta_3 + l_4\eta_4 + m_1\eta_1 + m_2\eta_2 \right\}^2. \end{aligned} \tag{2.23}$$

where α_1, b_1, b_2, b_3, and b_4 are arbitrary positive constants. From the Rellich identity [6, 7], it follows that

$$r \int_{\omega(r)} \left\{ \left(\frac{\partial w}{\partial s}\right)^2 - \left(\frac{\partial w}{\partial r}\right) \right\} ds = 2 \iint_{D_r} r\,\frac{\partial w}{\partial r}\,\Delta w\,dx\,dy + 2 \int_{\Sigma'} r\,\frac{\partial r}{\partial \sigma}\,\frac{\partial w}{\partial \sigma}\,\frac{\partial w}{\partial n}\,d\sigma - \int_{\Sigma'} r\,\frac{\partial r}{\partial n} |\mathrm{grad}\, w|^2 d\sigma. \tag{2.24}$$

Thus, for any $\delta_1 0$,

$$\int_{\omega(r)} \left\{ \left(\frac{\partial w}{\partial s}\right)^2 - \left(\frac{\partial w}{\partial r}\right)^2 \right\} ds \geq -\frac{\delta_1}{r} A_r(w) - \frac{r}{\delta_1}\eta_3 - \sqrt{2}\,\eta_2. \tag{2.25}$$

If $\partial^2 U/(\partial r^{-q})^2$ is to be non-negative, then the coefficient of the term $A_r(w) \int_{\omega(r)} w^2 ds$ must be non-negative. After insertion of (2.25) in (2.23) we find that this coefficient is positive if

$$q - p + 1 - \delta_1 \geq 0. \tag{2.26}$$

The coefficients of all other terms may be made positive by choosing C_1 sufficiently large provided

$$p>1. \tag{2.27}$$

But (2.26) and (2.27) are just the imposed conditions (2.17). Hence for such a choice of the constants, U is a convex function of r^{-q}.

We now let

$$\delta^*=[(R_1/\gamma_0)^{-q}-1]/[(R_1/\gamma_0)^{-q}-1]. \tag{2.28}$$

It follows then that

$$e^{Cr^{-2q}}E(r,w)\leq[e^{CR_1^{-2q}}E(R_1,w)]^{1-\delta^*}[e^{C\gamma_0^{-2q}}E(r_0,w)]^{\delta^*}. \tag{2.29}$$

Using (2.20) we obtain the inequality

$$\frac{1}{2}r^{-p}\int_{\omega(r)}w^2ds+\frac{(p-1)}{2}\int w^2r^{-(p+1)}dxdy+m_1\eta_1+m_2\eta_2+l_3\eta_3+l_4\eta_4$$
$$\leq C_1M_1^{1-\delta^*}\{m_1\eta_1+m_2\eta_2\}^{\delta^*}+C_2M_2^{1-\delta'}\{\kappa_1\epsilon_1+\kappa_2\epsilon_2\}^{\delta'}, \tag{2.30}$$

where the constants C_1 and C_2 are determined as indicated in [5]. We have used (2.10) or (2.12) for η_3 and the notation

$$\delta'=\delta\delta^*. \tag{2.31}$$

By use of Schwarz's inequality and inequalities (2.7), (2.9), and (2.30), equation (2.13) then yields the desired pointwise bounds for $w(P)$ at any point P in D_r.

In order to obtain the desired bounds in $D-D_r$ we first form from (2.13) $w_{,i}$, Δw and $\Delta w_{,i}$. For P a finite distance from the boundary of D_r the differentiation may be carried out under the integral sign on the right and all of the resulting integrals converge. Thus using (2.7), (2.9), (2.13), (2.29) we obtain the pointwise bounds in D_r not only for $w(P)$, but also for the quantities |grad w|, Δw and grad Δw (in fact estimates for any derivative of w). If we now integrate these inequalities over a portion of a spherical surface $\omega(r_1)$, where $r_1<r$ and each point on $\omega(r_1)$ is a finite distance from Σ, then we may regard $\omega(r_1)$ as a new Cauchy surface and proceed to define a new region D_{r_1}, continuing as indicated in [5]. We are thus able to obtain pointwise bounds for w and its derivatives at any point on the interior of D.

3. Analogous Viscous Flow Problems

Instead of interpreting our biharmonic Cauchy problem as a plate problem we could also have considered v in (1) to be the stream function in the slow flow of an incompressible viscous fluid [4, p. 541] (here $p\equiv0$). However, in this problem the actual value of v is of no physical significance. The velocity components are expressed in terms of derivatives of v, and the physical quantities which we measure on Σ are the velocity components, the vorticity and the pressure gradient. In terms of v the vorticity is the Laplacian of v and the pressure gradient involves the derivatives of Δv.

Hence we are interested in estimating only the derivatives of v and not v itself. At the same time we would like for these bounds to involve only η_2, ϵ_1, and ϵ_2, and not η_1. In order to accomplish this we consider instead of (2.13) the identity

$$w(P)-w(\overline{P})=-\int_{\omega(r)}[w-w(\overline{P})]\frac{\partial}{\partial n}(\Delta\Gamma)ds-\int_{\omega(r)}\Delta w\frac{\partial\Gamma}{\partial n}ds$$
$$-\int_{\Sigma'}\left\{[w-w(\overline{P})]\frac{\partial}{\partial n}(\Delta\Gamma)-\frac{\partial w}{\partial n}\Delta P+\Delta w\frac{\partial\Gamma}{\partial n}-\frac{\partial}{\partial n}(\Delta w)\Gamma\right\}d\sigma, \tag{3.1}$$

where $\overline{P}$ is the point on Σ at which $K(r_0)$ is tangent to C.

For points away from the boundary of D_r we differentiate (3.1) carrying out the differentiation of Γ on the right. An application of Schwarz's inequality yields the same types of terms as those encountered previously except for the terms $\int_{\omega(r)}[w-w(\overline{P})^2]ds$ and $\int_\Sigma [w-w(\overline{P})]^2 d\sigma$. Using the arguments employed in deriving (2.29) with w replaced by $w-w(\overline{P})$, we may bound the integral over $\omega(r)$ in terms of η_2, ϵ_1, ϵ_2, and $\int_\Sigma [w-w(\overline{P})]^2 d\sigma$. Let us now define λ by

$$\lambda=\min_{\psi}\frac{\int_0^l\left(\frac{\partial\psi}{\partial\sigma}\right)^2 d\sigma}{\int_0^l \psi^2 d\sigma} \tag{3.2}$$

among continuously differentiable functions ψ which vanish at the origin. The number λ is the lowest eigenvalue for a vibrating string fixed at the origin and free at $\sigma=l$. Thus

$$\lambda=\pi^2/4l^2. \tag{3.3}$$

We now let l be the shortest distance on Σ from $\overline{P}$ to either of the points of intersection of $\omega(r)$ and Σ, and denote by L the length of Σ. Thus

$$l\leq L/2 \tag{3.4}$$

and since $w-w(\overline{P})$ is clearly an admissible function ψ in (2.32)—the origin being taken at $\overline{P}$—we obtain using (3.3) and (3.4)

$$\int_\Sigma [w-w(\overline{P})]^2 d\sigma\leq L^2\pi^{-2}\int_\Sigma\left(\frac{\partial w}{\partial\sigma}\right)^2 d\sigma\leq L^2\pi^{-2}\eta_2. \tag{3.5}$$

It is thus possible to obtain in D_r the following bounds for the velocity components $w_{,x}$ using (2.7), (2.9), (2.30), (3.5), and (2.13):

$$|w_{,x}w_{,x}|^2\leq C_1'' M_1^{1-\delta^*}\eta_2^{\delta^*}+C_2'' M_2^{1-\delta'}\{\kappa_1\epsilon_1+\kappa_2\epsilon_2\}^{\delta'}, \tag{3.6}$$

where C_1'' and C_2'' are constants and all other terms are the same as those occurring on the right in (2.30). Note that as in [5] the constant M_1, which is a bound for $|w-w(\overline{P})|$, may be replaced by a constant L_1, which involves a bound for the kinetic energy in D.

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(Paper 65B2–55)

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(Including Papers in Outside Journals)

Selected Abstracts

On Stokes flow about a torus, W. H. Pell and L. E. Payne, *Mathematika* **7,** *78–92 (1960).*

This is the third of a series of papers on the Stokes (slow viscous) flow about axially symmetric bodies immersed in a uniform flow parallel to the axis of symmetry. The flow itself then has axial symmetry and is amenable to attack by the generalized axially symmetric potential theory of A. Weinstein. In this particular paper the flow in multiple connected regions is considered and, in particular, the problem of flow about a torus is solved. The mathematical problem which characterizes the flow does not have a unique solution. Uniqueness is insured by the physical condition that the pressure be continuous throughout the flow field.

Modern theories of materials, C. Truesdell, *Trans. Soc. Rheology* **IV,** *9–22 (1960).*

Theoretical work on rheological materials shows three distinct phases:

1. Early work, to 1945. Simple models, mainly one-dimensional and linear, are constructed by analogy to discrete systems. Various coefficients are named.

2. Intermediate period, 1945–1955. Ideal materials exhibiting nonlinear response are defined explicitly as exact mathematical realizations of a simple idea. Their properties are studied by means of general theorems and special exact solutions.

3. Current phase, from 1955. The most general constitutive equations consistent with the observed phenomena are determined.

Work of the second and third kinds is summarized, with particular reference to the different roles of methods of invariance.

Open problems are stated.

Graphs of bivariate normal probabilities, M. Zelen and N. C. Severo, *Ann. Math. Stat.* **31,** *No. 3, 619–624 (Sept. 1960).*

Recently there has been much activity dealing with the tabulation of the bivariate normal probability integral. D. B. Owen [3], [4] has summarized many of the properties of the bivariate normal distribution function and tabulated an auxiliary function which enables one to calculate the bivariate normal probability integral. In addition, the National Bureau of Standards [1] has compiled extensive tables of the bivariate normal integral drawn from the works of K. Pearson, Evelyn Fix and J. Neyman, and H. H. Germond. In this same volume, D. B. Owen has contributed an extensive section on applications.

It is the purpose of this paper to present three charts, which will enable one to easily compute the bivariate normal integral to a maximum error of 10^{-2}. This should be sufficient for most practical applications. Owen and Wiesen [5] have also presented charts with a similar objective; however, as pointed out below, we believe the charts presented here lend themselves more easily to visual interpolation. Actually the motivation for these charts came from the Owen and Wiesen work.

Comment on a paper of Mori on time-correlation expressions for transport properties, M. S. Green, *Phys. Rev.* **119,** *No. 3, 829–830 (Aug. 1960).*

An auto-correlation expression given by Mori for the thermal conductivity of a fluid is shown to be only apparently different from an expression previously derived by the author.

The Stokes flow about a spindle, W. H. Pell and L. E. Payne, Quart. Appl. Math. 18, 257–262 (1960).

This is a continuation of earlier work by the same authors. The methods of the generalized axially symmetric potential theory of A. Weinstein and certain representation theorems of L. Payne for the solution of repeated operator equations are applied to the solution of the Stokes (slow) flow about a spindle-shaped body. The stream function of the flow is found and an expression is given for the drag of the body.

Irrational power series, M. Newman, *Proc. Am. Math. Soc.* **11,** *699–802 (Oct. 1960).*

It is shown that if α is a real number, g a non-constant polynomial, then

$$\sum_{n=0}^{\infty} g([n\alpha])\,x^n$$

is a rational function of x if and only if α is a rational number. The same statement is proved for the function

$$\sum_{n=0}^{\infty} x^{[n\alpha]}.$$

Topological derivation of the Mayer density series for the pressure of an imperfect gas, M. S. Green, *J. Math. Phys.* **1,** *No. 5, 391–394 (Sept.–Oct. 1960).*

A new derivation of Mayer's classical density expansion for the pressure of an imperfect gas based on a classification of cluster graphs according to topological criteria is presented. The classification is a generalization of the classification of simple trees into trees with centers and trees with bicenters.

Ensemble method in the theory of irreversibility, R. Zwanzig, *J. Chem. Phys.* **33,** *No. 5, 1338–1341 (Nov. 1960).*

We describe a new formulation of methods introduced in the theory of irreversibility by Van Hove and Prigogine, with the purpose of making their ideas easier to understand and to apply. The main tool in this reformulation is the use of projection operators in the Hilbert space of Gibbsian ensemble densities. Projection operators are used to separate an ensemble density into a "relevant" part, needed for the calculation of mean values of specified observables, and the remaining "irrelevant" part. The relevant part is shown to satisfy a kinetic equation which is a generalization of Van Hove's "master equation to general order." Diagram summation methods are not used. The formalism is illustrated by a new derivation of the Prigogine-Brout master equation for a classical weakly interacting system.

On the theory of the critical point of a simple fluid, M. S. Green, *J. Chem. Phys.* **33,** *No. 5, 1403–1409 (Nov. 1950).*

The consequences of a new system of integral equations for the theory of the critical point are discussed. Reasons are given for believing that the fundamental assumption of the Ornstein-Zernicke theory about the direct correlation function is incorrect.

List of Titles

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Mass spectra of some deuteroethanes, E. I. Quinn and F. L. Mohler.

Heats of hydrolysis and formation of potassium borohydride, W. H. Johnson, R. H. Schumm, I. H. Wilson, and E. J. Prosen.

Heat of combustion of borazine $B_3N_3H_6$, M. V. Kilday, W. H. Johnson, and E. J. Prosen.
Thermodynamic properties of thorium dioxide from 298 to 1,200° K, A. C. Victor and T. B. Douglas.
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Vitrons as flow units in alkali silicate binary glasses, L. W. Tilton.
Tetragermanates of strontium, lead, and barium of formula type AB_4O_9, C. R. Robbins and E. M. Levin.

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Temperature dependence of elastic constants of some cermet specimens, S. Spinner.
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A bolometer mount efficiency measurement technique, G. F. Engen.
Telescope for measurement of optic angle of mica, S. Ruthberg.
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